Poincaré Inequalities and Moment Maps

Bo'az Klartag*

Abstract

We propose a new method for obtaining Poincaré-type inequalities on arbitrary convex bodies in $\mathbb{R}^n$. Our technique involves a dual version of Bochner's formula and a certain moment map, and it also applies to some non-convex sets. In particular, we generalize the central limit theorem for convex bodies to a class of non-convex domains, including the unit balls of ℓ_p-spaces in $\mathbb{R}^n$ for $0 < p < 1$.

1 Introduction

An important observation that goes back to Sudakov [22] and to Diaconis and Freedman [11] is that approximately gaussian marginals are intimately related to *thin shell inequalities*. That is, let X be a random vector in $\mathbb{R}^n$ with mean zero and identity covariance, where the dimension n is assumed very high. Suppose that X satisfies a thin shell inequality, of the form

$$\mathbb{E}\left(\frac{|X|^2}{n} - 1\right)^2 \ll 1. \tag{1}$$

It then follows that there are plenty of vectors $\theta \in \mathbb{R}^n$ for which the scalar product $\langle X, \theta\rangle$ is approximately a gaussian random variable. See von Weizsäcker [25], Bobkov [6], Anttila, Ball and Perissinaki [3] or [16, 18] for further explanations, and Eldan and Klartag [12] for connections to the hyperplane conjecture.

In this paper, Poincaré-type inequalities refer to inequalities in which the variance of a function is bounded in terms of an integral of a quadratic form involving the gradient of the function. One of the methods used to prove a thin shell bound such as (1) goes through such Poincaré-type inequalities in high-dimensional spaces. This approach was pursued in [17], where the Bochner formula was applied to study optimal thin shell bounds and Poincaré-type inequalities for the uniform measure on high-dimensional convex bodies. The technique in [17] and in the related work by Barthe and Cordero-Erausquin [5] relied very much on symmetries of the probability distribution under consideration. The method seemed quite irrelevant for

*School of Mathematical Sciences, Tel-Aviv University, Tel Aviv 69978, Israel. Supported in part by the Israel Science Foundation and by a Marie Curie Reintegration Grant from the Commission of the European Communities.
Email: klartagb@tau.ac.il

arbitrary convex bodies, possessing no symmetries. The following twist is proposed here: Introduce additional symmetries by considering a certain transportation of measure from a space of twice or thrice the dimension. The plan is to apply Bochner's formula in this higher dimensional space, and deduce a Poincaré-type inequality for the original measure.

We proceed by demonstrating the Poincaré-type inequalities that are obtained in the simplest case, perhaps, in which the convex set we investigate is $\mathbb{R}^n_+$, the orthant of all $x \in \mathbb{R}^n$ with positive coordinates. A function $\varphi : \mathbb{R}^n_+ \to (-\infty, \infty]$ is called p-convex, for $0 < p \leq 1$, if the function

$$(x_1, \ldots, x_n) \mapsto \varphi\left(x_1^{1/p}, \ldots, x_n^{1/p}\right)$$

is convex on $\mathbb{R}^n_+$. For instance $\varphi(x) = \sum_{i=1}^n \sqrt{x_i}$ is p-convex for any $p \leq 1/2$.

Theorem 1.1 *Let $n \geq 1, k > 1$ be integers. Suppose that μ is a Borel measure on $\mathbb{R}^n_+$ with density* $\exp(-\varphi)$*, where $\varphi : \mathbb{R}^n_+ \to (-\infty, \infty]$ is p-convex for $p = 1/k$. Assume that $f : \mathbb{R}^n_+ \to \mathbb{R}$ is a μ-integrable, locally Lipschitz function with $\int f d\mu = 0$. Then,*

$$\int_{\mathbb{R}^n_+} f^2 d\mu \leq \frac{k^2}{k-1} \sum_{i=1}^n \int_{\mathbb{R}^n_+} x_i^2 \left|\partial^i f(x)\right|^2 d\mu(x). \tag{2}$$

Here, $\partial^i f = \partial f / \partial x_i$ stands for the derivative of f with respect to the i^{th} variable.

We emphasize that the function f in Theorem 1.1 is not assumed to satisfy any boundary conditions. Compare, for example, to the Hardy-type inequalities in Matskewich and Sobolevskii [19]. We say that a subset $K \subset \mathbb{R}^n_+$ is p-convex for $0 < p \leq 1$, if

$$\left\{(x_1^p, \ldots, x_n^p) \; ; \; (x_1, \ldots, x_n) \in K\right\}$$

is a convex set. In other words, K is p-convex when the function that equals 0 on K and equals $+\infty$ outside K is p-convex. Observe that the intersection of p-convex sets is again a p-convex set. Dilations centered at the origin preserve p-convexity. For $p \neq 1$, translations do not necessarily preserve p-convexity, but p-convexity is preserved by translations conjugated with the map $x \mapsto (x_1^p, \ldots, x_n^p)$. From Theorem 1.1 we immediately deduce:

Corollary 1.2 *Let $n \geq 1, \ell > 1$ be integers, and assume that $K \subset \mathbb{R}^n_+$ is a $(1/\ell)$-convex set with a non-empty interior. Then, for any locally Lipschitz, integrable function $f : K \to \mathbb{R}$ with $\int_K f = 0$,*

$$\int_K f^2 dx \leq \frac{\ell^2}{\ell - 1} \sum_{i=1}^n \int_K x_i^2 \left|\partial^i f(x)\right|^2 dx.$$

For $x, y \in \mathbb{R}^n_+$ we write $x \leq y$ when $x_i \leq y_i$ for $i = 1, \ldots, n$. A function $\varphi : \mathbb{R}^n_+ \to (-\infty, \infty]$ is *increasing* when

$$x \leq y \qquad \Longrightarrow \qquad \varphi(x) \leq \varphi(y) \qquad\qquad (\text{for } x, y \in \mathbb{R}^n_+).$$

It is simple to see that when f is increasing and p-convex, it is also q-convex for any $0 < q < p$. A convex function is obviously 1-convex. A function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is unconditional if

$$\varphi(x_1, \ldots, x_n) = \varphi(|x_1|, \ldots, |x_n|) \qquad (x \in \mathbb{R}^n).$$

Observe that when φ is an unconditional, convex function on $\mathbb{R}^n$, the restriction $\varphi|_{\mathbb{R}^n_+}$ is necessarily increasing and p-convex for any $0 < p \le 1$. Thus Corollary 1.2 recovers the Poincaré-type inequalities from [17]: Quite unexpectedly, the unconditionality is used only to infer that when $\varphi|_{\mathbb{R}^n_+}$ is 1-convex, it is also $(1/2)$-convex. Theorem 1.1 may be generalized to measures on $\mathbb{R}^n$ whose density is unconditional, as follows:

Theorem 1.3 *Let μ be a probability measure on $\mathbb{R}^n$ with density* $\exp(-\varphi)$*, where $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is unconditional, and $\varphi|_{\mathbb{R}^n_+}$ is increasing and $1/k$-convex for an integer $k > 1$. Denote*

$$V_i = \int_{\mathbb{R}^n} x_i^2 d\mu(x) \qquad (i = 1, \ldots, n).$$

Then, for any μ-integrable, locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with $\int f d\mu = 0$,

$$\int_{\mathbb{R}^n} f^2 d\mu \le \int_{\mathbb{R}^n} \sum_{i=1}^n \left(\frac{k^2}{k-1} x_i^2 + V_i \right) \left| \partial^i f(x) \right|^2 d\mu(x). \tag{3}$$

Furthermore, when the function f is unconditional, we may eliminate the V_i's on the right-hand side of (3).

For $0 < p < 1$, denote by μ_p the uniform probability measure on the non-convex set

$$B_p^n = \left\{ x \in \mathbb{R}^n \, ; \, \sum_{i=1}^n |x_i|^p \le 1 \right\}.$$

Theorem 1.3 applies for the measure μ_p, with $k = \lceil 1/p \rceil$. Substituting $f(x) = |x|^2 - \int |y|^2 d\mu(y)$ into Theorem 1.3 yields thin shell bounds, which may be used to infer the existence of approximately gaussian marginals. Further discussion of the central limit theorem for *fractionally-convex bodies,* such as those in Theorem 1.3, is deferred to a future work. Once Theorem 1.1 and Corollary 1.2 are formulated, one is tempted to try and find a more direct proof of these inequalities. In Section 6 we discuss such a direct argument, based on the Brascamp-Lieb inequality [7], and obtain generalizations of Theorem 1.1 and Theorem 1.3 in which $k > 1$ is not necessarily an integer. Similarly, $\ell > 1$ does not have to be an integer in Corollary 1.2.

Next, suppose $K \subset \mathbb{R}^n$ is a convex body, i.e., a bounded, open convex set. We turn to the details of the Poincaré-type inequalities that are obtained for K. Recall that a function on $\mathbb{R}^n$ is log-concave if it takes the form $\exp(-H)$ for a convex function $H : \mathbb{R}^n \to (-\infty, \infty]$. A Borel measure on $\mathbb{R}^n$ is log-concave if its density is log-concave, and in particular, the uniform probability measure on an open, convex set is log-concave. We say that a smooth, convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ induces a "log-concave transportation to K" if the following two conditions hold:

(a) The function $\rho_\psi(x) = \det \nabla^2\psi(x)$ is positive and log-concave on $\mathbb{R}^n$, where $\nabla^2\psi$ is the Hessian of ψ.

(b) We have $\nabla\psi(\mathbb{R}^n) = K$, where $\nabla\psi(\mathbb{R}^n) = \{\nabla\psi(x); x \in \mathbb{R}^n\}$.

Observe that the map $x \mapsto \nabla\psi(x)$ pushes forward the measure whose density is ρ_ψ, to the uniform measure on the convex body K. For a given convex body $K \subset \mathbb{R}^n$, there are plenty of convex functions ψ that induce a log-concave transportation to K. In fact, for any log-concave function ρ on $\mathbb{R}^n$ whose integral equals the volume of K, there exists a convex function ψ which satisfies (a) and (b) with $\rho_\psi = \rho$. This follows from the general theory of optimal transportation of measure (e.g., Villani [24]). For indices $i, j, k = 1, \dots, n$ we abbreviate

$$\psi_i = \frac{\partial\psi}{\partial x_i}, \quad \psi_{ij} = \frac{\partial^2\psi}{\partial x_i \partial x_j}, \quad \psi_{ijk} = \frac{\partial^3\psi}{\partial x_i \partial x_j \partial x_k}.$$

We also write $\left(\psi^{ij}\right)_{i,j=1,\dots,n}$ for the inverse matrix to the Hessian matrix $\nabla^2\psi = (\psi_{ij})_{i,j=1,\dots,n}$. The Legendre transform of ψ is the function $\psi^* : K \to \mathbb{R}$ defined via

$$\psi^*(x) = \sup_{y \in \mathbb{R}^n} \left[\langle x, y\rangle - \psi(y)\right].$$

Then $\nabla\psi^*$ is the inverse map to $\nabla\psi$. With any $x \in K$ we associate the quadratic form $Q^*_{\psi,x}$ on $\mathbb{R}^n$ defined by

$$Q^*_{\psi,x}(V) = \sum_{i,j,k,\ell,m,p=1}^{n} V^i V^j \psi^{\ell m} \psi_{jkm} \psi^{kp} \psi_{i\ell p}$$

where $V = (V^1, \dots, V^n) \in \mathbb{R}^n$ and where the functions $\psi_{ij}, \psi^{\ell m}, \psi_{jkm}$ etc. are evaluated at the point $\nabla\psi^*(x)$. For $x \in K$ and $U \in \mathbb{R}^n$, set

$$Q_{\psi,x}(U) = \sup\left\{ 4\left(\sum_{i,j=1}^{n} \psi_{ij} U^i V^j\right)^2 ;\ V \in \mathbb{R}^n,\ Q^*_{\psi,x}(V) \le 1 \right\},$$

where ψ_{ij} is evaluated at the point $\nabla\psi^*(x)$. It could occur that $Q_{\psi,x}(U)$ is finite only for U in a certain subspace $E \subset \mathbb{R}^n$. Note that $Q_{\psi,x}$ is a quadratic form on that subspace E.

There is one technical assumption that we must make. In Section 3 we define the notion of *regularity at infinity* of the function ψ, and throughout the analysis below we conveniently assume the ψ is indeed regular at infinity. This assumption seems to hold in the examples that we consider. In the case where $K \subset \mathbb{R}^n$ is a simple rational polytope, regularity at infinity was investigated by Abreu [2], who explained that it holds under fairly mild assumptions.

Theorem 1.4 *Let $K \subset \mathbb{R}^n$ be a convex body. Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}$ induces a log-concave transportation to K. Assume further that ψ is regular at infinity. Then, for any Lipschitz function $f : K \to \mathbb{R}$,*

$$\int_K f = 0 \qquad \Rightarrow \qquad \int_K f^2 \le \int_K Q_{\psi,x}(\nabla f(x))\,dx.$$

In order to apply Theorem 1.4. one needs to select a function ψ which induces a log-concave transportation to K. Unfortunately, we are currently unaware of a general method for constructing a "reasonable" function ψ that satisfies (a) and (b), with good control over derivatives up to order three. In simple cases, such as when $K \subset \mathbb{R}^n$ is the cube or the simplex, Theorem 1.4 does yield meaningful inequalities. See Section 4 for a detailed analysis of the case of the simplex. In particular, Theorem 4.5 below provides somewhat unusual Poincaré-type inequalities for a class of distributions on the regular simplex. We present the proof of Theorem 1.1 in Section 2, before dealing with the more general Theorem 1.4 in Section 3. In Section 5 we prove Theorem 1.3. Throughout this paper, by a *smooth function* we mean a C^∞-smooth one.

Acknowledgements. Thanks to Semyon Alesker, Franck Barthe, Dmitry Faifman, Uri Grupel, Greg Kuperberg, Emanuel Milman, Yaron Ostrover, Leonid Polterovich, Yanir Rubinstein and Mikhail Sodin for interesting related discussions.

2 Non-Linear Measure Projection

In this section we prove Theorem 1.1. The analysis in this section is also intended to serve as a preparation for Section 3. Let $n, k \geq 1$ be positive integers, fixed throughout this section. Denote $m = nk$. We use

$$z = (z_1, \dots, z_n) \in (\mathbb{R}^k)^n = \mathbb{R}^{kn}$$

as coordinates in $\mathbb{R}^{kn}$, where $z_1, \dots, z_n$ are k-dimensional vectors. Consider the map $\pi : \mathbb{R}^m \to \overline{\mathbb{R}^n_+}$ defined by

$$\pi(z) = (|z_1|^k, \dots, |z_n|^k) \qquad\qquad (z_1, \dots, z_n) \in (\mathbb{R}^k)^n.$$

Here, $\overline{\mathbb{R}^n_+}$ is the closure of $\mathbb{R}^n_+$ in $\mathbb{R}^n$, and $|z_i|$ stands for the standard Euclidean norm of $z_i \in \mathbb{R}^k$. The continuous map π is proper, meaning that $\pi^{-1}(K)$ is compact whenever $K \subset \overline{\mathbb{R}^n_+}$ is compact. Let $S^{k-1} = \{y \in \mathbb{R}^k; |y| = 1\}$ denote the unit sphere in $\mathbb{R}^k$, and more generally, let $S^{k-1}(R) = \{y \in \mathbb{R}^k; |y| = R\}$. We write σ_R for the uniform probability measure on the sphere $S^{k-1}(R)$. With any $x \in \mathbb{R}^n_+$ we associate the cartesian product of spheres,

$$\pi^{-1}(x) := S^{k-1}(x_1^{1/k}) \times S^{k-1}(x_2^{1/k}) \times \ldots \times S^{k-1}(x_n^{1/k}) \subseteq (\mathbb{R}^k)^n = \mathbb{R}^m.$$

We denote by σ_x the uniform probability measure on $\pi^{-1}(x)$, that is, the direct product of the uniform probability measures on the spheres $S^{k-1}(x_j^{1/k})$ for $j = 1, \dots, n$.

We view the map π as a kind of *moment map*. The case $k = 2$ fits very well with the standard terminology, as in this case π is related to the moment map associated with the symplectic action of the group $(SO(2))^n$ on $(\mathbb{R}^2)^n$ (see, e.g., Cannas da Silva [9]). In the following lemma we verify that indeed the uniform measure on $\mathbb{R}^m$ is pushed forward to the uniform measure on $\mathbb{R}^n_+$ via the map π, up to a normalizing coefficient. We write Vol_k for the standard k-dimensional volume measure.

Lemma 2.1 *For any integrable function* $f : \mathbb{R}^n_+ \to \mathbb{R}$,

$$\int_{\mathbb{R}^m} f(\pi(z)) d\mathrm{Vol}_m(z) = \omega_{n,k} \int_{\mathbb{R}^n_+} f(x) d\mathrm{Vol}_n(x) \tag{4}$$

where $\omega_{n,k} = \left(\pi^{k/2}/\Gamma(k/2+1)\right)^n$ *is the* n^{th} *power of the volume of the* k*-dimensional unit ball. Furthermore, for any Borel set* $A \subseteq \mathbb{R}^m$,

$$\mathrm{Vol}_m(A) = \omega_{n,k} \int_{\mathbb{R}^n_+} \sigma_x(A) d\mathrm{Vol}_n(x). \tag{5}$$

Proof: Integrating in polar coordinates for each $z_j \in \mathbb{R}^k$ $(j = 1, \dots, n)$, we find that

$$\int_{\mathbb{R}^m} f(|z_1|^k, \dots, |z_n|^k) dz_1 \dots dz_n = \omega_k^n \int_{\mathbb{R}^n_+} f(x_1^k, \dots, x_n^k) \left(\prod_{j=1}^n x_j^{k-1} \right) dx_1 \dots dx_n,$$

where $\omega_k = k\pi^{k/2}/\Gamma(k/2+1)$ is the surface area of the unit sphere in $\mathbb{R}^k$. Applying the change of variables $(t_1, \dots, t_n) = (x_1^k, \dots, x_n^k)$ we obtain

$$\int_{\mathbb{R}^n_+} f(x_1^k, \dots, x_n^k) \left(\prod_{j=1}^n x_j^{k-1} \right) dx_1 \dots dx_n = k^{-n} \int_{\mathbb{R}^n_+} f(t_1, \dots, t_n) dt_1 \dots dt_n$$

and (4) follows. The relation (5) is proven in a similar fashion. □

Suppose ν is a Borel measure on $\mathbb{R}^m$. For a function $f \in L^2(\nu)$ we define

$$\|f\|_{H^{-1}(\nu)} = \sup \left\{ \int_{\mathbb{R}^m} fg d\nu \, ; \int_{\mathbb{R}^m} |\nabla g|^2 d\nu \le 1 \right\}, \tag{6}$$

where the supremum runs over all smooth functions $g : \mathbb{R}^m \to \mathbb{R}$ that belong to $L^2(\nu)$. Note that $\|f\|_{H^{-1}(\nu)} = +\infty$ when $\int f d\nu \neq 0$. The square of the $H^{-1}(\nu)$-norm is sub-additive in ν, as will be proven next:

Lemma 2.2 *Suppose* ν *is a Borel measure on* $\mathbb{R}^m$ *that takes the form*

$$\nu = \int_\Omega \nu_\alpha d\lambda(\alpha) \tag{7}$$

for Borel measures $\{\nu_\alpha\}_{\alpha \in \Omega}$ *on* $\mathbb{R}^m$ *and a measure* λ *on* Ω. *Then, for any* $f \in L^2(\nu)$,

$$\|f\|^2_{H^{-1}(\nu)} \le \int_\Omega \|f\|^2_{H^{-1}(\nu_\alpha)} d\lambda(\alpha).$$

Proof: Let g be a smooth function on $\mathbb{R}^m$ which belongs to $L^2(\nu)$. Since $f, g \in L^2(\nu_\alpha)$ for λ-almost any $\alpha \in \Omega$, then

$$\left| \int_{\mathbb{R}^m} fg d\nu_\alpha \right| \leq \|f\|_{H^{-1}(\nu_\alpha)} \sqrt{\int_{\mathbb{R}^m} |\nabla g|^2 \, d\nu_\alpha}$$

for λ-almost any $\alpha \in \Omega$. From (7) and the Cauchy-Schwartz inequality,

$$\begin{aligned} \left| \int_{\mathbb{R}^m} fg d\nu \right| &\leq \int_\Omega \|f\|_{H^{-1}(\nu_\alpha)} \left(\int_{\mathbb{R}^m} |\nabla g|^2 \, d\nu_\alpha \right)^{1/2} d\lambda(\alpha) \\ &\leq \sqrt{\int_{\mathbb{R}^m} \|f\|^2_{H^{-1}(\nu_\alpha)} d\lambda(\alpha)} \cdot \sqrt{\int_{\mathbb{R}^m} |\nabla g|^2 d\nu}. \end{aligned}$$

□

Recall that we use $(z_1, \ldots, z_n) \in (\mathbb{R}^k)^n$ as coordinates in $\mathbb{R}^m = \mathbb{R}^{kn}$. Let us furthermore denote $z_\ell = (z_\ell^1, \ldots, z_\ell^k) \in \mathbb{R}^k$, for any $\ell = 1, \ldots, n$.

Lemma 2.3 *Assume* $k \geq 2$. *Let* $x \in \mathbb{R}^n_+$. *Let* $1 \leq \ell \leq n, 1 \leq j \leq k$, *and denote* $f(z) = z_\ell^j$ *for* $z \in \mathbb{R}^m$. *Then,*

$$\|f\|_{H^{-1}(\sigma_x)} \leq \frac{x_\ell^{2/k}}{\sqrt{k(k-1)}}.$$

Proof: We claim that for any smooth function $h : \mathbb{R}^k \to \mathbb{R}$ and $\theta \in S^{k-1}$,

$$\int_{S^{k-1}} \langle y, \theta \rangle \, h(y) d\sigma_1(y) \leq \sqrt{\frac{1}{k(k-1)}} \cdot \sqrt{\int_{S^{k-1}} |\nabla h|^2 d\sigma_1}. \tag{8}$$

Indeed, (8) simply expresses the standard fact that $y \mapsto \sqrt{k}(y \cdot \theta)$ is a normalized eigenfunction of the Laplace-Beltrami operator on S^{k-1}, corresponding to the eigenvalue $k-1$ (see, e.g., Müller [20]). By scaling, we see that for any $R > 0$ and $\theta \in S^{k-1}$,

$$\int_{S^{k-1}(R)} \langle y, \theta \rangle \, h(y) d\sigma_R(y) \leq \frac{R^2}{\sqrt{k(k-1)}} \cdot \sqrt{\int_{S^{k-1}(R)} |\nabla h|^2 d\sigma_R}. \tag{9}$$

According to (9), for any fixed $z_1, \ldots, z_{\ell-1}, z_{\ell+1}, \ldots, z_n \in \mathbb{R}^k$ and a smooth function $g : \mathbb{R}^m \to \mathbb{R}$,

$$\int_{S^{k-1}(R_\ell)} z_\ell^j g(z_1, \ldots, z_n) d\sigma_{R_\ell}(z_\ell) \leq \frac{x_\ell^{2/k}}{\sqrt{k(k-1)}} \sqrt{\int_{S^{k-1}(R_\ell)} |\nabla g(z)|^2 d\sigma_{R_\ell}(z_\ell)},$$

where $R_\ell = x_\ell^{1/k}$. Recall that the probability measure σ_x is a product measure, and that σ_{R_ℓ} is the ℓ^{th} factor in this product. Integrating with respect to the remaining variables $z_1, \ldots, z_{\ell-1}, z_{\ell+1}, \ldots, z_n$, and using the Cauchy-Schwartz inequality, we obtain

$$\int_{\pi^{-1}(x)} z_\ell^j g(z) d\sigma_x(z) \leq \frac{x_\ell^{2/k}}{\sqrt{k(k-1)}} \sqrt{\int_{\pi^{-1}(x)} |\nabla g(z)|^2 d\sigma_x(z)}.$$

The lemma follows from the definition of the $H^{-1}(\sigma_x)$-norm. $\square$

The following lemma is one of the reasons for considering the higher-dimensional space $\mathbb{R}^m$, rather than working in the original space $\mathbb{R}^n_+$. The extra dimensions translate to "extra symmetries", which substitute for the explicit symmetries assumed in [17, Corollary 5] and in Barthe and Cordero-Erausquin [5, Section 3]. This effect actually seems more prominent in Section 3.

Lemma 2.4 *Assume* $k \geq 2$*, let* $1 \leq \ell \leq n, 1 \leq j \leq k$ *and let* $x \in \mathbb{R}^n_+$*. Suppose that* $f : \mathbb{R}^n_+ \to \mathbb{R}$ *is differentiable at* x*. Denote* $g(z) = f(\pi(z))$ *for* $z \in \mathbb{R}^m$*. Then,*

$$\left\| \frac{\partial g}{\partial z_\ell^j} \right\|_{H^{-1}(\sigma_x)} \leq \sqrt{\frac{k}{k-1}} \cdot x_\ell \left| \partial^\ell f(x) \right|.$$

Proof: Note that for $z \in \pi^{-1}(x)$,

$$\frac{\partial g}{\partial z_\ell^j}(z_1, \ldots, z_n) = k|z_\ell|^{k-2} z_\ell^j \cdot \partial^\ell f(|z_1|^k, \ldots, |z_n|^k) = \left(k x_\ell^{(k-2)/k} \partial^\ell f(x_1, \ldots, x_n) \right) z_\ell^j.$$

That is, the function $\partial g / \partial z_\ell^j$ is proportional to the linear function $z \mapsto z_\ell^j$ on the support of σ_x, and the proportion coefficient is exactly $k x_\ell^{(k-2)/k} \partial^\ell f(x_1, \ldots, x_n)$. According to Lemma 2.3,

$$\begin{aligned} \left\| \frac{\partial g}{\partial z_\ell^j} \right\|_{H^{-1}(\sigma_x)} &= k x_\ell^{(k-2)/k} \left| \partial^\ell f(x_1, \ldots, x_n) \right| \cdot \left\| z_\ell^j \right\|_{H^{-1}(\sigma_x)} \\ &\leq k x_\ell^{(k-2)/k} \left| \partial^\ell f(x_1, \ldots, x_n) \right| \cdot \frac{x_\ell^{2/k}}{\sqrt{k(k-1)}}. \end{aligned}$$

$\square$

Suppose $\Omega \subset \mathbb{R}^m$ is a bounded, open set. We say that a smooth function $u : \Omega \to \mathbb{R}$ is *smooth up to the boundary* if all of its derivatives of all orders are bounded in Ω. Note that when u is smooth up to the boundary, the boundary values of u and its derivatives are well-defined on $\partial\Omega$, by continuity. For $R > 1$ denote

$$\Omega_R = \left\{ (z_1, \ldots, z_n) \in (\mathbb{R}^k)^n \, ; \, R^{-1} < |z_i| < R \text{ for } i = 1, \ldots, n \right\}.$$

We denote by $\partial_{reg}\Omega_R$ the regular part of the boundary $\partial\Omega_R$. That is,

$$\partial_{reg}\Omega_R = \left(\bigcup_{i=1}^n A_i^- \right) \cup \left(\bigcup_{i=1}^n A_i^+ \right)$$

where

$$A_i^{\pm} = \left\{ z \in (\mathbb{R}^k)^n \; ; \, \log|z_i| = \pm \log R, \;\; R^{-1} < |z_j| < R \text{ for all } j \neq i \right\}. \tag{10}$$

We write $\mathcal{D}_R$ for the collection of all functions $u : \Omega_R \to \mathbb{R}$, smooth up to the boundary, that satisfy Neumann's condition:

$$\langle (\nabla u)_i, z_i \rangle = 0 \qquad \text{for any } \ i = 1, \dots, n, \quad z \in A_i^{\pm}. \tag{11}$$

Here, $\nabla u = ((\nabla u)_1, \dots, (\nabla u)_n) \in (\mathbb{R}^k)^n$. Let $G = (O(k))^n$, where $O(k)$ is the group of all orthogonal transformations in $\mathbb{R}^k$. The group G acts on $\mathbb{R}^m = (\mathbb{R}^k)^n$, via

$$g.(z_1, \dots, z_n) = (g_1(z_1), \dots, g_n(z_n))$$

for $g = (g_1, \dots, g_n) \in G = O(k)^n$ and $z = (z_1, \dots, z_n) \in (\mathbb{R}^k)^n$. A subset $U \subseteq \mathbb{R}^m$ is G-*invariant* if $g.z \in U$ for any $z \in U$, $g \in G$. Suppose $U \subseteq \mathbb{R}^m$ is G-invariant and $f : U \to \mathbb{R}$. We say that f is G-invariant if

$$f(g.z) = f(z) \qquad \text{for } \ g \in G, \ z \in U.$$

We write $\pi^{-1}(\mathbb{R}^n_+)$ for the collection of all $z \in (\mathbb{R}^k)^n$ with $z_i \neq 0$ for all i. Assume that $\psi : \pi^{-1}(\mathbb{R}^n_+) \to \mathbb{R}$ is a smooth function, and denote by ν the measure on $\pi^{-1}(\mathbb{R}^n_+)$ whose density is $\exp(-\psi)$. For a smooth function $u : \pi^{-1}(\mathbb{R}^n_+) \to \mathbb{R}$ write

$$\triangle^\nu u = e^{\psi} div(e^{-\psi} \nabla u) = \triangle u - \langle \nabla \psi, \nabla u \rangle,$$

where div stands for the usual divergence operator in $\mathbb{R}^m$. Integrating by parts, we see that for any $u, f : \Omega_R \to \mathbb{R}$ that are smooth up to the boundary,

$$\int_{\Omega_R} \langle \nabla u, \nabla f \rangle d\nu = -\int_{\Omega_R} f\,(\triangle^\nu u)\, d\nu + \int_{\partial_{reg}\Omega_R} f \langle \nabla u, N \rangle e^{-\varphi},$$

where N is the outer unit normal. In particular, when $f : \Omega_R \to \mathbb{R}$ is smooth up to the boundary and $u \in \mathcal{D}_R$,

$$\int_{\Omega_R} \langle \nabla u, \nabla f \rangle d\nu = -\int_{\Omega_R} f\,(\triangle^\nu u)\, d\nu. \tag{12}$$

The well-known Bochner identity states that for any smooth function $u : \Omega_R \to \mathbb{R}$,

$$\frac{1}{2} \triangle^\nu |\nabla u|^2 = \langle \nabla u, \nabla(\triangle^\nu u) \rangle + \sum_{i=1}^m |\nabla \partial^i u|^2 + \left\langle (\nabla^2 \psi) \nabla u, \nabla u \right\rangle, \tag{13}$$

as may be verified directly.

Lemma 2.5 *Let* $R > 1$ *and let* $u \in \mathcal{D}_R$ *be a* G-*invariant function. Then,*

$$\int_{\Omega_R} |\triangle^\nu u|^2\, d\nu = \int_{\Omega_R} \sum_{i=1}^m |\nabla \partial^i u|^2 d\nu + \int_{\Omega_R} \left\langle (\nabla^2 \psi) \nabla u, \nabla u \right\rangle d\nu.$$

Proof: We integrate the identity (13) over Ω_R. From (12),

$$\frac{1}{2}\int_{\Omega_R} \triangle^\nu |\nabla u|^2 \, d\nu + \int_{\Omega_R} |\triangle^\nu u|^2 \, d\nu = \int_{\Omega_R} \sum_{i=1}^m |\nabla \partial^i u|^2 d\nu + \int_{\Omega_R} \left\langle (\nabla^2 \psi) \nabla u, \nabla u \right\rangle d\nu,$$

since $u \in \mathcal{D}_R$. To conclude the lemma, it suffices to show that

$$\int_{\Omega_R} \triangle^\nu |\nabla u|^2 \, d\nu = 0.$$

This would follow from (12) once we show that $|\nabla u|^2 \in \mathcal{D}_R$. Hence, in order to conclude the lemma, we need to prove that

$$\left\langle \left(\nabla |\nabla u|^2 \right)_i, z_i \right\rangle = 0 \qquad \text{for any } \ i = 1, \ldots, n, \quad z \in A_i^\pm. \tag{14}$$

So far we did not apply the G-invariance of u. It will play a role in the proof of (14). Fix $i = 1, \ldots, n$. Since $u \in \mathcal{D}_R$, then according to (11), for $z \in A_i^\pm$,

$$\langle (\nabla u)_i, z_i \rangle = 0.$$

However, since u is G-invariant, then $(\nabla u)_i$ is always a vector proportional to z_i. We conclude that

$$(\nabla u)_i = 0 \ \text{ on } \ A_i^\pm. \tag{15}$$

We may differentiate (15) in the direction of ∇u, since ∇u is tangential to $\partial_{reg}\Omega_R$, and obtain

$$\left((\nabla^2 u) \nabla u \right)_i = 0 \ \text{ on } \ A_i^\pm. \tag{16}$$

Observe that

$$\nabla |\nabla u|^2 = 2(\nabla^2 u) \nabla u. \tag{17}$$

From (16) and (17) we deduce (14). $\square$

Lemma 2.6 *Suppose that $\varphi : \mathbb{R}^n_+ \to \mathbb{R}$ is smooth, and that the function*

$$(x_1, \ldots, x_n) \mapsto \varphi(x_1^k, \ldots, x_n^k)$$

is convex in $\mathbb{R}^n_+$. For $z \in \pi^{-1}(\mathbb{R}^n_+)$ denote $\psi(z) = \varphi(\pi(z))$. Then, for any G-invariant function $u : \mathbb{R}^m \to \mathbb{R}$,

$$\left\langle (\nabla^2 \psi) \nabla u, \nabla u \right\rangle \geq 0 \tag{18}$$

at any point $z \in \pi^{-1}(\mathbb{R}^n_+)$ in which u is differentiable.

Proof: Fix a point $z = (z_1, \ldots, z_n) \in (\mathbb{R}^k)^n$ with $z_i \neq 0$ for all i. Then the function

$$\mathbb{R}^n_+ \ni (a_1, \ldots, a_n) \mapsto \psi(a_1 z_1, \ldots, a_n z_n) \in \mathbb{R}$$

is convex on $\mathbb{R}^n_+$, by our assumption. In particular, $\nabla^2 \psi(z)|_E$ is positive semi-definite, where

$$E = \{(a_1 z_1, \ldots, a_n z_n) \ ; \ a_1, \ldots, a_n \in \mathbb{R}\} \subset \mathbb{R}^m$$

is an n-dimensional subspace. Since u is G-invariant and differentiable at z, then $\nabla u(z) \in E$, and (18) follows. $\square$

Write ν_R for the restriction of ν to Ω_R. We will use the following well-known fact from the theory of strongly elliptic operators on convex domains:

Lemma 2.7 *Suppose* $R > 1$. *Let* $f : \Omega_R \to \mathbb{R}$ *be a* G*-invariant function that is smooth up to the boundary with* $\int f d\nu_R = 0$. *Then, there exists a* G*-invariant function* $u \in \mathcal{D}_R$ *with* $\int u d\nu_R = 0$ *such that*

$$\triangle^\nu u = f \qquad \text{in } \Omega_R. \tag{19}$$

Proof sketch: Denote $Q_R = [-1/R, R]^n \subset \mathbb{R}^n$ and $g(|z_1|, \ldots, |z_n|) = f(z_1, \ldots, z_n)$ for $z \in \Omega_R$. Then g is smooth up to the boundary in Q_R. Denote by η the finite Borel measure on Q_R which is the push-forward of the measure ν_R under the map $(z_1, \ldots, z_n) \mapsto (|z_1|, \ldots, |z_n|)$. Then η has a density of the form $\exp(-\theta)$ on Q_R, where θ is smooth up to the boundary. Furthermore, $\int g d\eta = 0$. The task of solving (19) is reduced to the task of finding $u : Q_R \to \mathbb{R}$, smooth up to the boundary with $\int u d\eta = 0$, such that

$$\triangle u = g + \langle \nabla u, \nabla \theta \rangle, \tag{20}$$

and such that u satisfies Neumann's boundary condition on ∂Q_R. First, with the help of a crude Poincaré inequality and the Riesz representation theorem, we find a weak solution. That is, we find u in the Sobolev space $H^1(Q_R) = W^{1,2}(Q_R)$ with $\int u d\eta = 0$ such that (20) holds in the sense that

$$\int_{Q_R} \langle \nabla u, \nabla h \rangle \, d\eta = -\int_{Q_R} g h d\eta \qquad \text{for any } h \in H^1(Q_R). \tag{21}$$

See, e.g., Brezis [8, Chapter 9] or Folland [14, Chapter 7] for further explanations. Since θ is smooth up to the boundary, then $u \in H^k$ implies $\langle \nabla u, \nabla \theta \rangle \in H^{k-1}$ for any $k \geq 1$. Furthermore, by expanding into Fourier series in the cube Q_R, one sees that $\triangle u \in H^k$ implies $u \in H^{k+2}$ for any $k \geq 0$. Therefore, for any $k \geq 0$, if $u \in H^k$ then from (20) also $\triangle u \in H^{k-1}$, and hence $u \in H^{k+1}$. Therefore $u \in H^k$ for all k, and u is smooth up to the boundary in Q_R. From (21) we deduce that

$$\int_{Q_R} h \left(\triangle u - g - \langle \nabla u, \nabla \theta \rangle \right) d\eta = \int_{\partial Q_R} h \langle \nabla u, N \rangle e^{-\theta}$$

for any function h that is smooth up to the boundary in Q_R. Here, N is the outer unit normal. This implies that (20) holds true in the classical sense, and that u satisfies Neumann's condition at ∂Q_R, as required. $\square$

Lemma 2.8 *Let φ be as in Lemma 2.6. Suppose that μ is a Borel measure on $\mathbb{R}^n_+$ with density* $\exp(-\varphi)$. *Then, for any locally Lipschitz function* $f \in L^2(\mu) \cap L^1(\mu)$,

$$\mathrm{Var}_\mu(f) \leq \frac{k^2}{k-1} \sum_{i=1}^n \int_{\mathbb{R}^n_+} x_i^2 \left| \partial^i f(x) \right|^2 d\mu(x). \tag{22}$$

Here, $\mathrm{Var}_\mu(f) = \int (f-E)^2 d\mu$, *where* $E \in \mathbb{R}$ *is such that* $\int (f - E) d\mu = 0$.

Proof: By a standard approximation argument (e.g., convolve f with a localized bump function), we may assume that f is smooth on $\mathbb{R}^n_+$. Denote $\psi(z) = \varphi(\pi(z))$ for $z \in \pi^{-1}(\mathbb{R}^n_+)$. Let ν be the measure on $\mathbb{R}^m$ whose density is

$$z \mapsto \omega_{n,k}^{-1} \exp(-\psi(\pi(z))) \qquad\qquad (z \in \pi^{-1}(\mathbb{R}^n_+))$$

where $\omega_{n,k}$ is as in Lemma 2.1. Then π pushes the measure ν forward to the measure μ, as we learn from Lemma 2.1, and in fact,

$$\nu = \int_{\mathbb{R}^n_+} \sigma_x d\mu(x). \tag{23}$$

Fix $R > 1$ and denote $g(z) = f(\pi(z))$. The function g is smooth up to the boundary in Ω_R. Let $E_R \in \mathbb{R}$ be such that $\int (g - E_R) d\nu_R = 0$. According to Lemma 2.7, there exists a G-invariant function $u \in \mathcal{D}_R$ with $\int u d\nu_R = 0$ such that $\triangle^\nu u = -(g - E_R)$. Lemma 2.5 and Lemma 2.6 imply that

$$\int_{\Omega_R} |\triangle^\nu u|^2 d\nu \geq \int_{\Omega_R} \sum_{i=1}^m |\nabla \partial^i u|^2 d\nu. \tag{24}$$

We repeat the duality argument from [17, Section 2]:

$$\begin{aligned} &\int (g - E_R)^2 d\nu_R \\ &= -\int g \triangle^\nu u d\nu_R = \sum_{i=1}^m \int \partial^i g \partial^i u d\nu_R \leq \sum_{i=1}^m \|\partial^i g\|_{H^{-1}(\nu_R)} \sqrt{\int |\nabla \partial^i u|^2 d\nu_R} \\ &\leq \sqrt{\sum_{i=1}^m \|\partial^i g\|^2_{H^{-1}(\nu_R)}} \sqrt{\int \sum_{i=1}^m |\nabla \partial^i u|^2 d\nu_R} \leq \sqrt{\sum_{i=1}^m \|\partial^i g\|^2_{H^{-1}(\nu_R)}} \sqrt{\int |\triangle^\nu u|^2 d\nu_R}, \end{aligned} \tag{25}$$

where we used (24) in the last inequality. Therefore,

$$\int_{\Omega_R} (g - E_R)^2 d\nu_R \leq \sum_{i=1}^m \|\partial^i g\|^2_{H^{-1}(\nu_R)} = \sum_{\ell=1}^n \sum_{j=1}^k \left\| \frac{\partial g}{\partial z_\ell^j} \right\|^2_{H^{-1}(\nu_R)}. \tag{26}$$

According to Lemma 2.2 and to (23), for any $\ell = 1, \dots, n$ and $j = 1, \dots, k$,

$$\left\| \frac{\partial g}{\partial z_\ell^j} \right\|^2_{H^{-1}(\nu_R)} \leq \int_{\mathbb{R}^n_+} \left\| \frac{\partial g}{\partial z_\ell^j} \right\|^2_{H^{-1}(\sigma_x)} d\mu(x) \leq \frac{k}{k-1} \int_{\mathbb{R}^n_+} x_\ell^2 \left| \partial^\ell f(x) \right|^2 d\mu(x), \tag{27}$$

where the last inequality is the content of Lemma 2.4. By combining (26) and (27), and letting R tend to infinity, we obtain

$$\mathrm{Var}_\mu(f) = \mathrm{Var}_\nu(g) \leq \frac{k^2}{k-1} \sum_{i=1}^n \int_{\mathbb{R}^n_+} x_i^2 \left| \partial^i f(x) \right|^2 d\mu(x).$$

$\square$

Proof of Theorem 1.1: Assume first that φ is finite and smooth. All we need in order to deduce (2) from (22) is to remove the assumption that $f \in L^2(\mu)$. To that end, given a locally Lipschitz $f \in L^1(\mu)$ and $M > 0$, we consider the truncation

$$f_M = \max\{\min\{f, M\}, -M\}.$$

Then $f_M \in L^2(\mu)$ is locally Lipschitz. The set $E_M = \{x \in \mathbb{R}^n; |f(x)| = M\}$ is of measure zero for almost every $M > 0$, as $E_M \cap E_{\tilde{M}} = \emptyset$ for $M \neq \tilde{M}$. We apply (22) for f_M and let M tend to infinity, and obtain (2). This completes the proof in the case where φ is finite and smooth. For the general case, a standard approximation argument is needed. One possibility is to observe that it is enough to prove the theorem where the integrals over $\mathbb{R}^n_+$ are replaced by integrals over the cube

$$\left[R^{-1}, R\right]^n \subset \mathbb{R}^n_+,$$

for any $R > 1$. On the bounded cube, it is straightforward to approximate $\exp(-\varphi)$ by a finite, smooth density, such that both the left-hand side and the right-hand side of (2) are well-approximated, for a given locally Lipschitz function f. This completes the proof. $\square$

Remark 2.9 Suppose $k_1, \ldots, k_n \geq 2$ are integers, and that the function $\varphi : \mathbb{R}^n_+ \to (-\infty, \infty]$ is such that

$$(x_1, \ldots, x_n) \mapsto \varphi(x_1^{k_1}, \ldots, x_n^{k_n})$$

is convex on $\mathbb{R}^n_+$. It is straightforward to adapt the proof of Theorem 1.1 to this case. We obtain a variant of Theorem 1.1, in which the inequality (2) is modified as follows: The factor $k^2/(k-1)$ is inserted into the sum, and replaced by $k_i^2/(k_i - 1)$. See Theorem 6.1 below.

3 Toric Kähler Manifolds

This section provides a proof of Theorem 1.4. Throughout this section, we assume that we are given a convex body $K \subset \mathbb{R}^n$, and a smooth, convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ with $\nabla\psi(\mathbb{R}^n) = K$. Most of the argument generalizes to any open, convex set $K \subset \mathbb{R}^n$. In particular, the analysis in Section 2 for $k = 2$ is parallel to the case where K equals $\mathbb{R}^n_+$ and $\psi(x) = \sum_{i=1}^n \exp(x_i)$.

The proof of Theorem 1.4 is essentially an interpretation of the dual Bochner inequality in a certain toric Kähler manifold. We begin with a quick review of the the basic definitions, see e.g. Tian [23, Chapter 1] for more information. Suppose X is a complex manifold of complex

dimension n. The induced almost complex structure is a certain smooth map $J : TX \to TX$, such that for any $p \in X$ the restriction $J|_{T_pX}$ is a linear operator onto T_pX with

$$J^2|_{T_pX} = -I.$$

In fact, in an open set $U \subset \mathbb{C}^n$ containing the origin, consider the map $f(z) = \sqrt{-1}\,z$ defined in a neighborhood of zero. Its derivative at zero is $J|_{T_0U}$. One verifies that this construction of J does not depend on the choice of the chart, as the transition functions are holomorphic. A closed 2-form ω on X is *Kähler* if the bilinear form

$$g_\omega(u, v) = \omega(u, Jv) \qquad (p \in X, \quad u, v \in T_pX)$$

is a Riemannian metric, which is also J-invariant (i.e., $g_\omega(u, v) = g_\omega(Ju, Jv)$ for any $p \in X$ and $u, v \in T_pX$). Next, we specialize to the case of toric Kähler manifolds, see also Abreu [1] and Gromov [15]. We consider the complex torus

$$\mathbb{T}^n_{\mathbb{C}} = \mathbb{C}^n/(\sqrt{-1}\mathbb{Z}^n) = \left\{x + \sqrt{-1}y \,;\, x \in \mathbb{R}^n,\ y \in \mathbb{R}^n/\mathbb{Z}^n\right\}.$$

(Perhaps it is more common to say that $(\mathbb{C}^*)^n$ is the complex torus, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Note that $\exp(2\pi z)$ is a biholomorphism between $\mathbb{T}^1_{\mathbb{C}}$ and $\mathbb{C}^*$). The real torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ acts on the complex manifold $\mathbb{T}^n_{\mathbb{C}}$ via

$$t.(x + \sqrt{-1}y) = x + \sqrt{-1}(y + t) \qquad \left(t \in \mathbb{T}^n, x + \sqrt{-1}y \in \mathbb{T}^n_{\mathbb{C}}\right).$$

Functions, vector fields and differential forms on $\mathbb{R}^n$ have toric-invariant extensions to $\mathbb{T}^n_{\mathbb{C}}$. For instance, we extend the convex function ψ to $\mathbb{T}^n_{\mathbb{C}}$ by

$$\psi(x + \sqrt{-1}y) = \psi(x) \qquad \text{for } x + \sqrt{-1}y \in \mathbb{T}^n_{\mathbb{C}}.$$

Then ψ is a $\mathbb{T}^n$-invariant function on the complex manifold $\mathbb{T}^n_{\mathbb{C}}$. With a slight abuse of notation, we use the same letter to denote a function on $\mathbb{R}^n$, and its toric-invariant extension to $\mathbb{T}^n_{\mathbb{C}}$. Consider the Kähler form on $\mathbb{T}^n_{\mathbb{C}}$ defined by

$$\omega_\psi = 2\sqrt{-1}\partial\bar{\partial}\psi = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} \psi_{ij} dz_i \wedge d\bar{z}_j.$$

Abbreviating $g_\psi = g_{\omega_\psi}$, we have

$$g_\psi\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g_\psi\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \psi_{ij} \qquad (i, j = 1, \ldots, n)$$

while $g_\psi\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = 0$ for any i, j. Furthermore, observe that

$$\omega_\psi^n = \rho_\psi \mathrm{Vol}_{2n}$$

where Vol_{2n} is the standard volume form on $\mathbb{T}^n_{\mathbb{C}}$ and $\rho_\psi(x) = \det \nabla^2\psi(x)$ for $x \in \mathbb{R}^n$. It is customary to call the map $x + \sqrt{-1}y \mapsto \nabla\psi(x)$ the associated *moment map*, see Abreu [1] and Gromov [15].

Below we review in great detail some of the standard formulae of Riemannian geometry in the case of a toric Kähler manifold. As much as possible, we prefer real formulae in real variables. One reason for this is that the complex notation fits well only with the case $k = 2$ in Section 2. For a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ we write

$$\nabla^\psi u = \sum_{i,j=1}^n \psi^{ij} u_i \frac{\partial}{\partial x_j} = \sum_{j=1}^n u^j \frac{\partial}{\partial x_j}$$

for the Riemannian gradient of u, where we abbreviate $u^j = \sum_{i=1}^n \psi^{ij} u_i$. Next, we describe the connection ∇^ψ that corresponds to the Riemannian metric g_ψ. As is computed, e.g., in Tian [23],

$$\nabla^\psi_{\frac{\partial}{\partial y_j}} \frac{\partial}{\partial x_k} = \frac{1}{2}\sum_{\ell=1}^n \psi^\ell_{jk} \frac{\partial}{\partial y_\ell}, \qquad \nabla^\psi_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = \frac{1}{2}\sum_{\ell=1}^n \psi^\ell_{jk} \frac{\partial}{\partial x_\ell}$$

where $\psi^\ell_{jk} = \sum_{m=1}^n \psi^{\ell m}\psi_{jkm}$. We view the Hessian $\nabla^{\psi,2}h$ of a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ as a linear operator on T_pX, specifically,

$$T_pX \ni U \mapsto \nabla^\psi_U \nabla^\psi h \in T_pX.$$

In coordinates, for a smooth function $h : \mathbb{R}^n \to \mathbb{R}$,

$$\nabla^{\psi,2}h\left(\frac{\partial}{\partial x_i}\right) = \sum_{j,k=1}^n \left(\psi^{jk}h_{ik} - \frac{1}{2}\psi^{jk}_i h_k\right)\frac{\partial}{\partial x_j},$$

$$\nabla^{\psi,2}h\left(\frac{\partial}{\partial y_i}\right) = \frac{1}{2}\sum_{j,k=1}^n \psi^{jk}_i h_k \frac{\partial}{\partial y_j}, \tag{28}$$

where $\psi^{jk}_i = \sum_{\ell,m=1}^n \psi^{\ell j}\psi^{mk}\psi_{i\ell m}$. It is unfortunate that we have to work with the real Hessian, and not with the simpler complex Hessian. We denote by $\triangle^\psi$ the Riemmanian Laplacian on $\mathbb{T}^n_{\mathbb{C}}$, corresponding to the Riemmanian metric g_ψ. Then $\triangle^\psi h$ is the trace of $\nabla^{\psi,2}h$, and for a smooth function $h : \mathbb{R}^n \to \mathbb{R}$,

$$\triangle^\psi h = \sum_{i,j=1}^n \psi^{ij}h_{ij}.$$

The Bochner-Weitzenböck formula from Riemannian geometry (e.g. Petersen [21, Section 7.3.1]) states that for any smooth function $u : \mathbb{R}^n \to \mathbb{R}$,

$$\frac{1}{2}\triangle^\psi|\nabla^\psi u|^2 = \langle \nabla^\psi u, \nabla^\psi(\triangle^\psi u)\rangle + |\nabla^{\psi,2}u|^2_{HS} + \mathrm{Ric}_\psi(\nabla^\psi u, \nabla^\psi u) \tag{29}$$

where $|\nabla^{\psi,2}u|^2_{HS}$ is the Hilbert-Schmidt norm of the Hessian, and where Ric_ψ is the Ricci form, which is the bilinear form given by

$$\mathrm{Ric}_\psi\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = -\frac{1}{2}\frac{\partial^2 \log \rho_\psi}{\partial x_j \partial x_k}$$

for $j, k = 1, \dots, n$. Note that $\mathrm{Ric}_\psi(\nabla^\psi u, \nabla^\psi u) \geq 0$ when ρ_ψ is log-concave.

Definition 3.1 *Suppose (M, g) is a Riemannian manifold, ∇ is the standard Levi-Civita connection, and ν a Borel measure on M. Let V be a vector field on M, which is locally ν-integrable. We set*

$$\|V\|_{H^{-1}(\nu)} = \sup\left\{\int_M \langle V, \nabla h\rangle d\nu\,;\ \int_M |\nabla^2 h|^2_{HS} d\nu \leq 1\right\} \tag{30}$$

where the supremum runs over all smooth functions $h : M \to \mathbb{R}$ such that $\langle V, \nabla h\rangle$ is ν-integrable.

The proof of Lemma 2.2 immediately generalizes to

$$\nu = \int_\Omega \nu_\alpha d\lambda(\alpha) \qquad \Rightarrow \qquad \|V\|^2_{H^{-1}(\nu)} \leq \int_\Omega \|V\|^2_{H^{-1}(\nu_\alpha)} d\lambda(\alpha). \tag{31}$$

Next, we use the $\mathbb{T}^n$-invariance and obtain a lower bound for $|\nabla^{\psi,2}u|^2_{HS}$ in terms of the first derivatives of u. Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. Denote by $E_p \subset T_p X$ the subspace spanned by $\frac{\partial}{\partial y_j}$ $(j = 1, \dots, n)$. As in any Riemannian manifold, the operator $\nabla^{\psi,2}u$ is symmetric with respect to the Riemmannian metric g_ψ. Furthermore, from (28) we learn that E_p is an invariant subspace of the operator $\nabla^{\psi,2}u$, and the matrix representing the operator $\nabla^{\psi,2}u|_{E_p}$ in the basis $\frac{\partial}{\partial y_k}$ $(k = 1, \dots, n)$ is

$$\left(\frac{1}{2}\sum_{j=1}^n u^j \psi^\ell_{jk}\right)_{k,\ell=1,\dots,n}.$$

Consequently,

$$\begin{aligned} \left|\nabla^{\psi,2}u\right|^2_{HS} &\geq \left|\left(\nabla^{\psi,2}u\Big|_{E_p}\right)\right|^2_{HS} = \mathrm{Trace}\left[\left(\nabla^{\psi,2}u\Big|_{E_p}\right)^2\right] \\ &= \frac{1}{4}\sum_{i,j,m,p=1}^n u^i u^j \psi^p_{jm}\psi^m_{ip}. \end{aligned} \tag{32}$$

For $x \in \mathbb{R}^n$ we denote by σ_x the uniform probability measure on the real torus $\{x + \sqrt{-1}y\,;\, y \in \mathbb{T}^n\}$. For a vector field $U = \sum_{i=1}^n U^i \frac{\partial}{\partial x_i}$ set

$$\tilde{Q}_{\psi,x}(U) = \sup\left\{\left(\sum_{j=1}^n \psi_{ij} U^j V^j\right)^2 ;\ \frac{1}{4}\sum_{i,j,k,\ell,m,p=1}^n V^i V^j \psi^{\ell m}\psi_{jkm}\psi^{kp}\psi_{i\ell p} \leq 1\right\},$$

where the supremum runs over all $V^1, \dots, V^n \in \mathbb{R}^n$. Here, $\psi^{\ell m}, \psi_{jkm}$ etc. are evaluated at x. Observe that $\tilde{Q}_{\psi,x}$ is essentially the same quadratic form as $Q_{\psi,\nabla\psi(x)}$ mentioned in the Introduction. That is, if $h = f(\nabla\psi(x))$, then

$$\tilde{Q}_{\psi,x}\left(\nabla^{\psi} h\right) = Q_{\psi,\nabla\psi(x)}(\nabla f).$$

Lemma 3.2 *Let* $u : \mathbb{R}^n \to \mathbb{R}$*. Then, for any* $x \in \mathbb{R}^n$ *in which* u *is differentiable,*

$$\|\nabla^{\psi} u\|^2_{H^{-1}(\sigma_x)} \leq \tilde{Q}_{\psi,x}(\nabla^{\psi} u).$$

Proof: The vector field $\nabla^{\psi} u$ on $\mathbb{T}^n_{\mathbb{C}}$ is $\mathbb{T}^n$-invariant. It therefore suffices to restrict our attention to $\mathbb{T}^n$-invariant functions h in the definition (30) of $\|\nabla^{\psi} u\|_{H^{-1}(\sigma_x)}$ (i.e., if h is not $\mathbb{T}^n$-invariant, then average it with respect to the $\mathbb{T}^n$-action). Suppose that $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. From (32),

$$\int_{\mathbb{T}^n_{\mathbb{C}}} |\nabla^{\psi,2} h|^2_{HS} d\sigma_x \geq \frac{1}{4} \sum_{i,j,k,\ell,m,p=1}^{n} h^i h^j \psi^{\ell m} \psi_{jkm} \psi^{kp} \psi_{i\ell p}$$

where the functions on the right-hand side are evaluated at the point x. Since

$$\int_{\mathbb{T}^{\mathbb{C}}_n} \langle \nabla^{\psi} u, \nabla^{\psi} h \rangle d\sigma_x = \sum_{i,j=1}^{n} \psi_{ij} u^i h^j,$$

the lemma follows from the definition of the H^{-1} norm. $\square$

Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a smooth function on $\mathbb{R}^n$, with $\inf \varphi > -\infty$. Consider the finite Borel measure μ on $\mathbb{T}^n_{\mathbb{C}}$ that is induced by the volume form $\exp(-\varphi)\omega^n_{\psi}$. That is, μ is the measure on $\mathbb{T}^n_{\mathbb{C}}$ whose density with respect to the standard Lebesgue measure on $\mathbb{T}^n_{\mathbb{C}}$ is

$$\exp(-\varphi(x))\rho_{\psi}(x).$$

Observe that

$$\mu = \int_{\mathbb{R}^n} \sigma_x e^{-\varphi(x)} \rho_{\psi}(x) dx. \tag{33}$$

For a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ denote

$$\triangle^{\mu} u = \triangle^{\psi} u - \sum_{i,j=1}^{n} \psi^{ij} u_i \varphi_j. \tag{34}$$

Integrating by parts, we see that when $u, h : \mathbb{R}^n \to \mathbb{R}$ are smooth functions, with at least one of them compactly-supported,

$$\int_{\mathbb{T}^n_{\mathbb{C}}} h(\triangle^{\mu} u) d\mu = -\int_{\mathbb{T}^n_{\mathbb{C}}} \langle \nabla^{\psi} u, \nabla^{\psi} h \rangle d\mu. \tag{35}$$

We assume that the following Bakry-Émery-Ricci condition holds true:

($\star$) For any $x \in \mathbb{R}^n$, the matrix

$$\left(\varphi_{i\ell} - \frac{1}{2} \sum_{k=1}^{n} \psi_{i\ell}^{k} \varphi_k - \frac{1}{2} \frac{\partial^2 \log \rho_\psi}{\partial x_i \partial x_\ell} \right)_{i,\ell=1,\ldots,n}$$

is positive semi-definite.

Condition ($\star$) is equivalent to the pointwise inequality,

$$\left\langle (\nabla^{\psi,2} \varphi) U, U \right\rangle + \mathrm{Ric}_\psi(U, U) \geq 0 \tag{36}$$

for any vector field of the form $U = \sum_{i=1}^{n} U^i \frac{\partial}{\partial x_i}$. In the terminology of Bakry and Émery [4], condition ($\star$) means that the Bakry-Émery-Ricci tensor (also known as Γ_2 or the "second carré du champ") is positive semi-definite, when restricted to the subspace spanned by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. The only case that is relevant for Theorem 1.4, is when ρ_ψ is log-concave and $\varphi \equiv 1$. Condition ($\star$) clearly holds true in this case. Theorem 1.1 is related to the case where $\psi(x) = \sum_{i=1}^{n} e^{x_i}$, and condition ($\star$) amounts to the convexity of the function $\varphi(2 \log x_1, \ldots, 2 \log x_n)$ in the interior of $\mathbb{R}^n_+$.

As explained in the Introduction, we have to impose certain restrictions on the behavior of ψ and φ at infinity. We say that the pair of functions (ψ, φ) is *regular at infinity* if there exists a linear space X of smooth functions $u : \mathbb{R}^n \to \mathbb{R}$ which has the following properties:

(a) For any $u, h \in X$ we have that $h \triangle^\mu u, \left\langle \nabla^\psi u, \nabla^\psi h \right\rangle \in L^1(\mu)$, and the the identity (35) holds true. The same holds also when $u \in X$, and $h : \mathbb{R}^n \to \mathbb{R}$ is such that $h(\nabla \psi^*(x))$ is a Lipschitz function on K.

(b) The constant functions belong to X. If $u \in X$, then also $\triangle^\mu u, |\nabla^\psi u|^2 \in X$.

(c) Denote by $\mathcal{H} \subset L^2(\mu)$ the subspace of all functions $f : \mathbb{R}^n \to \mathbb{R}$ with $\int f d\mu = 0$. Then the space

$$\{ \triangle^\mu u \, ; \, u \in X \}$$

is dense in $\mathcal{H}$ in the topology of $L^2(\mu)$.

We say that ψ is regular at infinity if $(\psi, 1)$ is regular at infinity. Observe that the space of compactly-supported, smooth functions might not satisfy (c), as there might exist non-constant, smooth functions $f \in L^2(\mu)$ with $\triangle^\mu f \equiv 0$. The space X is supposed to capture a sort of "Neumann's condition at infinity". A thorough investigation of regularity at infinity is beyond the scope of the present paper, which focuses on the Bochner method combined with additional symmetries in higher dimension.

Remark 3.3 Suppose that the Riemannian manifold $(\mathbb{T}^n_{\mathbb{C}}, g_\psi)$ admits a smooth compactification. That is, assume that $(\mathbb{T}^n_{\mathbb{C}}, g_\psi)$ embeds in a compact, smooth Riemannian manifold (M, g) as a dense subset of full measure, that the moment map $\nabla \psi$ extends to a smooth function on the entire M, and that the $\mathbb{T}^n$-action on $(\mathbb{T}^n_{\mathbb{C}}, g_\psi)$ extends to a $\mathbb{T}^n$-action on (M, g). In this case, ψ is regular at infinity: We may define X to be the restriction to $\mathbb{T}^n_{\mathbb{C}}$ of all $\mathbb{T}^n$-invariant, smooth functions on the compact Riemannian manifold M. Indeed, condition (b) then holds

trivially. As for condition (a), observe that h extends to a Lipschitz function on M as it is the composition of the Lipschitz maps $h(\nabla\psi^*)$ and $\nabla\psi$, hence integrations by parts of h against $\triangle^{\psi}u$ may be carried out in M. We conclude that condition (a) holds true since $\mathbb{T}^n_{\mathbb{C}}$ is of full measure in M, and the integrals in (35) are equivalent to integrals over the entire M. Condition (c) follows from the standard theory of elliptic partial differential equations on a compact, connected, smooth Riemannian manifold.

Remark 3.4 Another relevant type of compactification is related to the so-called orbifolds or V-manifolds, which are smooth manifolds except for some rather tame singularities. We refer the reader, e.g., to Chiang [10] for Harmonic analysis on Riemannian orbifolds. In particular, there is a notion of a smooth function on the entire orbifold, and the Laplace equation may be solved with smooth functions on compact orbifolds. We conclude that the function ψ is regular at infinity whenever $(\mathbb{T}^n_{\mathbb{C}}, g_\psi)$ embeds in a compact Riemannian orbifold as a dense subset of full measure, such that $\nabla\psi$ and the toric action extend smoothly to the entire Riemannian orbifold. In the case of K being a rational, simple polytope, all functions ψ admitting such embedding were characterized by Abreu [2]. He gave a clear criterion in terms of ψ^*, which seems to hold in most cases of interest. Since rational, simple polytopes are dense among convex bodies, one is tempted to conjecture that Abreu's mild condition for regularity at infinity may be generalized to the class of all convex bodies.

The following lemma is a well-known Bochner-type integration by parts formula. For completeness, we include its proof.

Lemma 3.5 *Assume that $(\star)$ holds true, and that (ψ, φ) is regular at infinity. Then for any $u \in X$,*

$$\int_{\mathbb{T}^n_{\mathbb{C}}} |\triangle^{\mu} u|^2 d\mu \geq \int_{\mathbb{T}^n_{\mathbb{C}}} |\nabla^{\psi,2} u|^2_{HS} d\mu.$$

Proof: From (29) and (34) we obtain the identity

$$(37)\quad \begin{aligned} &\frac{1}{2}\triangle^{\mu}|\nabla^{\psi}u|^2 \\ &= \langle \nabla^{\psi}u, \nabla^{\psi}(\triangle^{\mu}u)\rangle + |\nabla^{\psi,2}u|^2_{HS} + \mathrm{Ric}_{\psi}(\nabla^{\psi}u, \nabla^{\psi}u) + \left\langle \left(\nabla^{\psi,2}\varphi\right) \nabla^{\psi}u, \nabla^{\psi}u \right\rangle. \end{aligned}$$

From our assumption $(\star)$,

$$(38)\qquad \frac{1}{2}\triangle^{\mu}|\nabla^{\psi}u|^2 \geq \langle \nabla^{\psi}u, \nabla^{\psi}(\triangle^{\mu}u)\rangle + |\nabla^{\psi,2}u|^2_{HS}.$$

Integrating the above inequality over $\mathbb{T}^n_{\mathbb{C}}$, we obtain

$$0 \geq -\int_{\mathbb{T}^n_{\mathbb{C}}} |\triangle^{\psi}u|^2 d\mu + \int_{\mathbb{T}^n_{\mathbb{C}}} |\nabla^{\psi,2}u|^2_{HS} d\mu,$$

since $\int_{\mathbb{T}^n_{\mathbb{C}}} (\triangle^{\psi}h) d\mu = 0$ for any $h \in X$. $\square$

Theorem 1.4 is the case $\varphi \equiv 1$ of the next proposition.

Proposition 3.6 *Let* $K \subset \mathbb{R}^n$ *be a convex body. Suppose that* $\psi, \varphi : \mathbb{R}^n \to \mathbb{R}$ *are smooth functions, such that* ψ *is convex with* $\det \nabla^2 \psi(x) > 0$ *for any* $x \in \mathbb{R}^n$*, and such that* $\inf \varphi > -\infty$*. Assume that* $\nabla\psi(\mathbb{R}^n) = K$*, that condition (*$\star$*) above holds true, and that* (ψ, φ) *is regular at infinity. Let* μ *be the measure (33) and denote by* ν *the finite Borel measure on* K *which is the push-forward of* μ *under* $\nabla\psi$*. Then, for any Lipschitz function* $f : K \to \mathbb{R}$*,*

$$\int_K f d\nu = 0 \qquad \Rightarrow \qquad \int_K f^2 d\nu \leq \int_K Q_{\psi,x}(\nabla f) d\nu. \tag{39}$$

Proof: We denote $h(x) = f(\nabla\psi(x))$. Let $u \in X$. With the help of Lemma 3.5, the duality argument (25) is replaced by

$$\begin{aligned} -\int_{\mathbb{T}^n_{\mathbb{C}}} h\,(\triangle^\mu u)\, d\mu &= \int_{\mathbb{T}^n_{\mathbb{C}}} \langle \nabla^\psi h, \nabla^\psi u \rangle d\mu \\ &\leq \|\nabla^\psi h\|_{H^{-1}(\mu)} \sqrt{\int_{\mathbb{T}^n_{\mathbb{C}}} |\nabla^{\psi,2} u|^2_{HS}\, d\mu} \leq \|\nabla^\psi h\|_{H^{-1}(\mu)} \sqrt{\int_{\mathbb{T}^n_{\mathbb{C}}} |\triangle^\mu u|^2 d\mu}. \end{aligned} \tag{40}$$

Since f is bounded, then also is h is bounded, hence $h \in L^2(\mu)$ with

$$\int_{\mathbb{T}^n_{\mathbb{C}}} h d\mu = \int_K f d\nu = 0.$$

Consequently, there exists $u_k \in X$ for $k = 1, 2, \ldots$ such that $\triangle^\mu u_k \to -h$ when $k \to \infty$, in the topology of $L^2(\mu)$. From (40),

$$\int_K f^2 d\nu = \int_{\mathbb{T}^n_{\mathbb{C}}} h^2 d\mu \leq \|\nabla^\psi h\|^2_{H^{-1}(\mu)}.$$

Combine the latter inequality with (31), (33) and Lemma 3.2, and obtain

$$\begin{aligned} \int_K f^2 d\nu \leq \|\nabla^\psi h\|^2_{H^{-1}(\mu)} &\leq \int_{\mathbb{R}^n} \|\nabla^\psi h\|^2_{H^{-1}(\sigma_x)} e^{-\varphi(x)} \rho_\psi(x) dx \\ &\leq \int_{\mathbb{R}^n} \tilde{Q}_{\psi,x}\left(\nabla^\psi h\right) e^{-\varphi(x)} \rho_\psi(x) dx = \int_K Q_{\psi,x}\,(\nabla f)\, d\nu(x). \end{aligned}$$

$\square$

Remark 3.7 In principle, one may formulate and prove Theorem 1.4 in terms of ψ^*, rather than going back and forth between ψ and ψ^*, or between $\mathbb{R}^n$ and K. The reason for preferring ψ, is that for $n > 1$, the condition that ψ induces a log-concave transportation for K appears simpler than the corresponding condition for ψ^*. On the other hand, for a convex function ψ in one variable, $\log \psi''$ is concave if and only if $1/(\psi^*)''$ is concave.

Remark 3.8 When (X, μ, d) is a metric measure space and $T : X \to Y$ is a locally Lipschitz map, we may trivially transfer any Poincaré type inequality on X to a Poincaré type inequality on Y. An example is given in Corollary 4.4 below, where a Poincaré type inequality for the simplex is deduced from the standard Poincaré inequality on $\mathbb{CP}^n$. Similarly, when $\rho_\psi = \exp(-|x|^2/2)$, we may, in principle, transfer the standard Poincaré inequality of the gaussian measure to an inequality on K. The approach that we promote in this paper, of using "dual Bochner in a higher dimension with extra symmetries", is different, and it seems to be applicable to situations in which the former method fails. Note that we do not assume any Poincaré-type inequality for the log-concave density ρ_ψ.

4 An Example: The Simplex

In order to demonstrate the potential of our paradigm, we present in this section the Poincaré-type inequalities that follow from Theorem 1.4 in the particular case of the simplex. We also discuss the inequalities that follow via the direct method outlined in Remark 3.8. Our first goal is to apply Theorem 1.4 in the setting where $K \subset \mathbb{R}^n$ is the open simplex whose vertices are $0, e_1, \ldots, e_n \in \mathbb{R}^n$. Here, $e_1, \ldots, e_n$ are the standard unit vectors in $\mathbb{R}^n$. Note that this simplex is not regular; Later, we will translate the results to the regular simplex. Consider the smooth, convex function,

$$\psi(x_1, \ldots, x_n) = \log\left(1 + e^{x_1} + \ldots + e^{x_n}\right) \qquad (x \in \mathbb{R}^n).$$

Note that

$$\nabla\psi(x) = \frac{(e^{x_1}, \ldots, e^{x_n})}{1 + e^{x_1} + \ldots + e^{x_n}}. \tag{41}$$

It is straightforward to verify from (41) that

$$\nabla\psi(\mathbb{R}^n) = K.$$

Our choice of ψ is motivated by the fact that the Kähler manifold $(\mathbb{T}^n_{\mathbb{C}}, \omega_\psi)$ is isometric, up to a normalization, to a dense open subset of full measure of the complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric, see e.g., the first pages of Tian [23] or Cannes da Silva [9] for more information. For instance, the Riemannian manifold $(\mathbb{T}^1_{\mathbb{C}}, g_\psi)$ is precisely the two-dimensional sphere of radius one, without the north and the south poles. The moment map $\nabla\psi$ and the toric action may be extended smoothly to $\mathbb{CP}^n$, and in view of Remark 3.3, we deduce that the function ψ is regular at infinity. We continue by computing the second derivatives,

$$\nabla^2\psi(x) = \left(\frac{e^{x_i}\delta_{ij}}{1 + e^{x_1} + \ldots + e^{x_n}} - \frac{e^{x_i + x_j}}{(1 + e^{x_1} + \ldots + e^{x_n})^2} \right)_{i,j=1,\ldots,n}.$$

Here, δ_{ij} is Kronecker's delta.

Lemma 4.1 *(a) The function*

$$x \mapsto \det \nabla^2\psi(x)$$

is log-concave in $\mathbb{R}^n$.

(b) The inverse hessian matrix is

$$\psi^{ij}(x) = \left(1 + \sum_{j=1}^{n} e^{x_j}\right)\left[1 + \delta_{ij} e^{-x_i}\right].$$

Proof: Denote

$$v = \frac{(e^{x_1}, \ldots, e^{x_n})}{1 + e^{x_1} + \ldots + e^{x_n}} \in \mathbb{R}^n.$$

We write

$$\nabla^2 \psi(x) = A - B,$$

where A is a diagonal matrix with v_i at the i^{th} diagonal entry, and $B = (v_i v_j)_{i,j=1,\ldots,n}$. The determinant of a rank-one perturbation has a simple formula:

$$\det \nabla^2 \psi(x) = \det(A - B) = \det(A)\left[1 - \langle A^{-1} v, v\rangle\right].$$

This boils down to

$$\det \nabla^2 \psi(x) = \exp\left(-(n+1)\psi(x) + \sum_{j=1}^{n} x_j\right), \tag{42}$$

which is log-concave as ψ is convex. It remains to prove (b). According to the Sherman-Morisson formula for the inverse of a rank-one perturbation,

$$\left(\nabla^2 \psi(x)\right)^{-1} = (A - B)^{-1} = A^{-1} + \frac{A^{-1} B A^{-1}}{1 - \langle A^{-1} v, v\rangle},$$

as may be verified directly. Equivalently,

$$\psi^{ij} = \left(1 + \sum_{j=1}^{n} e^{x_j}\right)\left[1 + \delta_{ij} e^{-x_i}\right].$$

□

Thus ψ induces a log-concave transportation to K. Note that $2Ric_\psi = (n+1)g_\psi$, as follows from (42). In particular, we have a very good uniform lower bound for the Ricci curvature, which implies a rather strong Poincaré inequality on $\mathbb{CP}^n$ – even a log-Sobolev inequality – according to Bakry and Émery [4]. Consequently, the simple, direct method of Remark 3.8 has the potential to produce interesting inequalities in the case of the simplex. Still, first we would like to test the applicability of Theorem 1.4 here, and to that end, we will write down explicit expressions for the formidable quadratic form $Q_{\psi,x}$. We compute that

$$\begin{aligned}\psi_{ijk} &= 2e^{x_i + x_j + x_k - 3\psi} + e^{x_i - \psi}\delta_{ij}\delta_{jk} \\ &\quad - \left[e^{x_j + x_k - 2\psi}\delta_{ij} + e^{x_i + x_j - 2\psi}\delta_{ik} + e^{x_i + x_k - 2\psi}\delta_{jk}\right].\end{aligned}$$

Therefore,

$$\psi_{jk}^{\ell} = \sum_{i=1}^{n} \psi^{i\ell}\psi_{ijk} = \delta_{jk}\delta_{j\ell} - \delta_{j\ell}e^{x_k-\psi} - \delta_{k\ell}e^{x_j-\psi}$$

and, for any fixed $i, j = 1, \dots, n$,

$$\sum_{k,\ell=1}^{n} \psi_{jk}^{\ell}\psi_{i\ell}^{k} = (n+3)e^{x_i+x_j-2\psi} - e^{x_i-\psi} - e^{x_j-\psi} + \delta_{ij}(1-2e^{x_i-\psi}).$$

Consequently,

$$\begin{aligned} Q^{*}_{\psi,\nabla\psi(x)}(V) &= \sum_{i,j=1}^{n} V^i V^j \left[(n+3)e^{x_i+x_j-2\psi} - e^{x_i-\psi} - e^{x_j-\psi} + \delta_{ij}(1-2e^{x_i-\psi})\right] \\ &= \sum_{i,j,k=1}^{n} \psi_{ij} a_k^i V^k V^j, \end{aligned}$$

where, for $i, k = 1, \dots, n$,

$$a_k^i = e^{x_k}\left(1 - e^{-x_i}\right) + \delta_{ik}\left(e^{\psi-x_i} - 2\right).$$

We are not confused by the minus signs, and we remember that $Q^{*}_{\psi,\nabla\psi(x)}$ must be a positive semi-definite quadratic form on $\mathbb{R}^n$. Consider for a moment the scalar product

$$(U, V) = \sum_{i,j=1}^{n} \psi_{ij} U^i V^j \qquad\qquad (U, V \in \mathbb{R}^n)$$

and the linear operator

$$A(U) = \left(\sum_{k=1}^{n} a_k^i U^k\right)_{i=1,\dots,n} \in \mathbb{R}^n \qquad \text{for } U = (U^1, \dots, U^n) \in \mathbb{R}^n.$$

Then A is symmetric with respect to the scalar product $(\cdot,\cdot)$, and $Q^{*}_{\psi,\nabla\psi(x)}(V) = (A(V), V)$ for $V \in \mathbb{R}^n$. Observe that

$$Q_{\psi,\nabla\psi(x)}(U) = \sup\left\{4(U,V)^2\ ;\ V \in \mathbb{R}^n,\ Q^{*}_{\psi,\nabla\psi(x)}(V) \le 1\right\} = 4\left(A^{-1}(U), U\right).$$

Denote $B = A^{-1} = \left(b_j^i\right)_{i,j=1,\dots,n}$. In order to compute the b_j^i's, we apply the Sherman-Morisson formula again, and obtain the expression

$$b_j^i = \frac{\delta_{ij}}{\psi_j^{-1} - 2} - \frac{\psi_j}{\psi_j^{-1} - 2} \cdot \frac{e^{\psi} - \psi_i^{-1}}{\psi_i^{-1} - 2}\left(1 + \sum_{k=1}^{n} \frac{e^{\psi}\psi_k - 1}{\psi_k^{-1} - 2}\right)^{-1}.$$

Therefore,

$$\sum_{\ell=1}^{n} \psi_{i\ell} b_j^{\ell} = \frac{\psi_i^2}{1-2\psi_i}\delta_{ij} + \frac{\psi_i^2}{1-2\psi_i} \cdot \frac{\psi_j^2}{1-2\psi_j} \cdot \frac{2 - e^{\psi}}{1 + \sum_{k=1}^{n}\left[(e^{\psi}\psi_k - 1)/(\psi_k^{-1} - 2)\right]}.$$

Finally, recalling that $\psi_i, \exp(\psi)$ are to be evaluated at the point $\nabla\psi^*(x) = (\nabla\psi)^{-1}x$, we obtain the positive semi-definite quadratic form

$$\frac{1}{4}Q_{\psi,x}(U) = \sum_{i=1}^{n} \frac{x_i^2|U^i|^2}{1-2x_i} - \left(\sum_{i=1}^{n} \frac{x_i^2 U^i}{1-2x_i}\right)^2 \left(\sum_{k=0}^{n} \frac{x_k^2}{1-2x_k}\right)^{-1} \tag{43}$$

where we define $x_0 = 1 - \sum_{j=1}^{n} x_j$. In conclusion, so far we have obtained the following:

Corollary 4.2 *Let $K \subset \mathbb{R}^n$ be the simplex which is the convex hull of $0, e_1, \ldots, e_n$, where $e_1, \ldots, e_n$ are the standard unit vectors in $\mathbb{R}^n$. Then for any Lipschitz function $f : K \to \mathbb{R}$ with $\int_K f = 0$,*

$$\int_K f^2(x)dx \le 4\int_K \left[\sum_{i=1}^{n} \frac{x_i^2\left|\partial^i f\right|^2}{1-2x_i} - \left(\sum_{k=0}^{n} \frac{x_k^2}{1-2x_k}\right)^{-1} \left(\sum_{i=1}^{n} \frac{x_i^2 \partial^i f}{1-2x_i}\right)^2\right] dx$$

where $x_0 = 1 - \sum_{k=1}^{n} x_k$.

Next, observe that Corollary 1.2 applies for the uniform measure on the simplex K, with $\ell = 2$. We are unaware of any advantage of Corollary 4.2 over the inequality that follows from Corollary 1.2 in this case. Yet, the importance of Corollary 4.2 to us is that it perhaps demonstrates that the very general Theorem 1.4 is not entirely inapplicable. We continue by translating our results to the regular simplex.

Recall that $\mathbb{R}_+^{n+1}$ is the orthant of all $x \in \mathbb{R}^{n+1}$ with positive coordinates. Consider the n-dimensional regular simplex

$$\triangle^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}_+^{n+1} \, ; \, \sum_{j=0}^{n} x_j = 1 \right\}. \tag{44}$$

Observe that the projection

$$(x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$$

is a measure preserving one-to-one correspondence between $\triangle^n$ and K. Let $p \in \triangle^n$, and suppose that $f : \triangle^n \to \mathbb{R}$ is differentiable at p. For indices $i, j = 0, \ldots, n$ we set

$$E^{ij} f(p) = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) f(p).$$

Observe that $E_{ij}f(p)$ is well-defined, since the vector field $\partial/\partial x_i - \partial/\partial x_j$ belongs to the tangent space $T_p\triangle^n$ for any $p \in \triangle^n$.

Theorem 4.3 *Let $\triangle^n$ be the simplex (44). Then for any Lipschitz function $f : \triangle^n \to \mathbb{R}$ with $\int_{\triangle^n} f = 0$,*

$$\int_{\triangle^n} f^2(x)dx \le 4\int_{\triangle^n} \left(\sum_{k=0}^{n} \frac{x_k^2}{1-2x_k}\right)^{-1} \sum_{i\neq j} \frac{x_i^2 x_j^2}{(1-2x_i)(1-2x_j)} \left|E^{ij} f\right|^2 dx.$$

Here, the sum runs over the $n(n+1)/2$ distinct pairs of indices $i, j \in \{0, \ldots, n\}$.

Proof: For $(x_0, \dots, x_n) \in \triangle^n$ denote

$$g(x_1, \dots, x_n) = f(x_0, \dots, x_n).$$

Then $g : K \to \mathbb{R}$ is a Lipschitz function. We compute that

$$Q_{\psi,x}(\nabla g(x_1, \dots, x_n)) = 4 \left(\sum_{k=0}^{n} \frac{x_k^2}{1 - 2x_k} \right)^{-1} \sum_{i \neq j} \frac{x_i^2 x_j^2}{(1 - 2x_i)(1 - 2x_j)} \left| E^{ij} f \right|^2$$

where $Q_{\psi,x}$ is given by (43). The theorem thus follows from Corollary 4.2. □

We would like to compare Theorem 4.3 with the push-forward of the usual Poincaré inequality on $\mathbb{CP}^n$ via the moment map. Recall that $S^{2n+1}(R) = \{z \in \mathbb{C}^{n+1}; \sum_{i=0}^n |z_i|^2 = R^2\}$ is the sphere of radius R in $\mathbb{C}^{n+1}$, equipped with the induced Riemannian metric. Recall that the Riemannian manifold $(\mathbb{T}^n_{\mathbb{C}}, g_\psi)$ is embedded in $\mathbb{CP}^n$ equipped with the Fubini-Study metric, up to some normalization. In fact, with respect to the normalization dictated by ψ, we may view the complex projective space $\mathbb{CP}^n$ as a quotient of the sphere $S^{2n+1}(2) \subset \mathbb{C}^{n+1}$ by a circle action. If we extend the map $\nabla\psi$ from $\mathbb{T}^n_{\mathbb{C}}$ to $\mathbb{CP}^n$ by continuity, and then lift it to a circle-invariant function on $S^{2n+1}(2)$, then we obtain the function

$$S^{2n+1}(2) \ni (z_0, \dots, z_n) \mapsto \left(\frac{|z_1|^2}{4}, \dots, \frac{|z_n|^2}{4} \right) \in K.$$

The manifold $\mathbb{CP}^n$ inherits the Poincaré inequality for even functions on the sphere $S^{2n+1}(2)$ (see, e.g., Müller [20] for the inequality on the sphere). Consequently, the standard Poincaré inequality on $\mathbb{CP}^n$ is the bound

$$\int_{\mathbb{R}^n} u(x)\rho_\psi(x)dx = 0 \quad \Rightarrow \quad \int_{\mathbb{R}^n} u^2(x)\rho_\psi(x)dx \leq \frac{1}{n+1} \int_{\mathbb{R}^n} |\nabla^\psi u(x)|^2 \rho_\psi(x)dx, \tag{45}$$

valid for any function $u : \mathbb{R}^n \to \mathbb{R}$ for which $x \mapsto u(\nabla\psi^*(x))$ is Lipschitz. (One way to make sure that indeed $n+1$ is the first non-zero eigenvalue of $-\triangle^\psi$, is to verify that equality in (45) is attained for the eigenfunction $u = \psi_1 - 1/(n+1)$.) Translating (45) to the simplex $K \subset \mathbb{R}^n$ via the moment map $\nabla\psi$, we obtain in a straightforward manner:

Corollary 4.4 *Let $K \subset \mathbb{R}^n$ be the simplex which is the convex hull of $0, e_1, \dots, e_n$, where $e_1, \dots, e_n$ are the standard unit vectors in $\mathbb{R}^n$. Then for any Lipschitz function $f : K \to \mathbb{R}$ with $\int_K f = 0$,*

$$\int_K f^2(x)dx \leq \frac{1}{n+1} \int_K \left[\sum_{i=1}^n x_i \left| \partial^i f \right|^2 - \left(\sum_{i=1}^n x_i \partial^i f \right)^2 \right] dx.$$

Equivalently, let $\triangle^n$ be the simplex (44). Then for any Lipschitz function $f : \triangle^n \to \mathbb{R}$,

$$\int_{\triangle^n} f = 0 \qquad \Rightarrow \qquad \int_{\triangle^n} f^2(x)dx \leq \frac{1}{n+1} \int_{\triangle^n} \sum_{i \neq j} x_i x_j \left| E^{ij} f \right|^2 dx. \tag{46}$$

Here, the sum runs over the $n(n+1)/2$ distinct pairs of indices $i, j \in \{0, \dots, n\}$.

Note that when the dimension n is high, for a random point $x \in K$ we typically have $x_i \approx \frac{1}{n}$. Therefore Corollary 4.4 is not so different from Corollary 4.2, when the dimension is high, while the latter is less elegant. Since Corollary 4.4 has a much shorter proof, then naïvely it seems that the general method suggested in Theorem 1.4 is not entirely essential in the case of the simplex. In a sense, when proving Corollary 4.2 we only used the fact that $\mathbb{CP}^n$ has a non-negative Ricci form, and we did not fully exploit the relatively high curvature of $\mathbb{CP}^n$. The picture is different once we use the freedom to select a suitable weight function $\exp(-\varphi)$ in Proposition 3.6. The following theorem provides a taste of the Poincaré-type inequalities on the simplex that follow from Proposition 3.6. Recall the notion of a p-convex function from the Introduction.

Theorem 4.5 *Let $\triangle^n$ be the simplex (44), let $q \geq 0$ and let $\varphi : \mathbb{R}_+^{n+1} \to \mathbb{R}$ be a $(1/2)$-convex function, smooth up to the boundary in $\triangle^n$, homogenous of degree q. Denote $M = \sup_{x \in \triangle^n} \varphi(x)$, and assume that*

$$Mq \leq n. \tag{47}$$

(Alternatively, we can assume condition (48) below in place of (47).) Denote by ν the finite Borel measure on $\triangle^n \subset \mathbb{R}^{n+1}$ whose density with respect to the Lebesgue measure on $\triangle^n$ is

$$(x_0, \dots, x_n) \mapsto \exp\left(-\varphi\left(x_0, \dots, x_n\right)\right) \qquad (x \in \triangle^n).$$

Then for any Lipschitz function $f : \triangle^n \to \mathbb{R}$ with $\int_{\triangle^n} f d\nu = 0$,

$$\int_{\triangle^n} f^2(x) d\nu(x) \leq 4 \int_{\triangle^n} \left(\sum_{k=0}^n \frac{x_k^2}{1 - 2x_k} \right)^{-1} \sum_{i \neq j} \frac{x_i^2 x_j^2}{(1-2x_i)(1-2x_j)} \left| E^{ij} f \right|^2 d\nu(x).$$

Here, the sum runs over the $n(n+1)/2$ distinct pairs of indices $i, j \in \{0, \dots, n\}$.

Proof: Note that φ extends by continuity to the closure $\overline{\mathbb{R}_+^{n+1}} \setminus \{0\}$. Define

$$f(z_0, \dots, z_n) = \varphi\left(\frac{|z_0|^2}{4}, \dots, \frac{|z_n|^2}{4} \right) \qquad (0 \neq z \in \mathbb{C}^{n+1}),$$

and observe that f is smooth on $S^{2n+1}(2)$ as φ is smooth up to the boundary in $\triangle^n$. For a point $p \in S^{2n+1}(2)$ we write $E_p \subset T_p(S^{2n+1}(2))$ for the subspace spanned by the gradients of the functions $|z_0|^2, \dots, |z_n|^2$ on $S^{2n+1}(2)$. Arguing as in Lemma 2.6, we see that

$$\left\langle (\nabla^2 f) U, U \right\rangle \geq 0 \quad \text{for any } \ p \in S^{2n+1}(2), U \in E_p.$$

From (47),

$$\left\langle (\nabla^2 f) U, U \right\rangle + \frac{n - qM}{2} |U|^2 \geq 0 \quad \text{for any } \ p \in S^{2n+1}(2), U \in E_p.$$

Since $f(p) \leq M$ for any $p \in S^{2n+1}(2)$, then f satisfies

$$\left\langle (\nabla^2 f) U, U \right\rangle + \frac{n - qf(p)}{2} |U|^2 \geq 0 \quad \text{for any } \ p \in S^{2n+1}(2), U \in E_p. \tag{48}$$

The remainder of the proof is devoted to showing that condition (48) suffices for the application of Proposition 3.6. To that end, denote by $\pi : S^{2n+1}(2) \to \mathbb{CP}^n$ the quotient map, which associates with any $z \in S^{2n+1}(2)$ the complex line through the origin that passes through z. Note that when $p \in S^{2n+1}(2)$ is such that $\pi(p) \in \mathbb{T}^n_{\mathbb{C}}$, the subspace $\pi_*(E_p)$ is the linear span of $\partial/\partial x_1, \ldots, \partial/\partial x_n$. We need to check that condition $(\star)$ from Section 3 holds true, and that the pair

$$\left(\psi(x), \varphi\left(\frac{(1, e^{x_1}, \ldots, e^{x_n})}{1 + e^{x_1} + \ldots + e^{x_n}}\right)\right)$$

is regular at infinity. The main observation here is that both requirements are satisfied when

$$\left\langle \left(\nabla^2_{S^{2n+1}(2)} f\right) U, U \right\rangle + \mathrm{Ric}_{S^{2n+1}(2)}(U, U) \geq 0 \quad \text{for any } \ p \in S^{2n+1}(2), U \in E_p. \tag{49}$$

Here, $\nabla^2_{S^{2n+1}(2)} f$ stands for the Hessian of f with respect to the Riemannian metric on $S^{2n+1}(2)$. Indeed, it is straightforward to verify that the Bakry-Émery-Ricci tensor of a smooth function $g : \mathbb{CP}^n \to \mathbb{R}$ is positive semi-definite on $\pi_*(E_p)$, if and only if the Bakry-Émery-Ricci tensor of $g \circ \pi : S^{2n+1}(2) \to \mathbb{R}$ is positive semi-definite on E_p. Hence (49) implies condition $(\star)$ from Section 3. The regularity at infinity is not an issue, as $f \circ \pi^{-1}$ is well-defined and smooth on the entire $\mathbb{CP}^n$. Since $\mathrm{Ric}_{S^{2n+1}(2)}(U, U) = n|U|^2/2$ and f is homogenous of degree $2q$, then (49) is equivalent to (48). The theorem is thus proven. $\square$

Remark 4.6 Observe that the Poincaré inequality on $\mathbb{CP}^n$, rendered as (45) above, essentially remains true when we replace the integrals over the entire $\mathbb{CP}^n$ with integrals over a geodesically-convex subset of $\mathbb{CP}^n$. This follows from the Bochner formula, with a slightly weaker constant $2/(n+1)$ in place of the factor $1/(n+1)$ from (45). See Escovar [13, Theorem 4.3] for details and for a better constant. Consequently, (46) remains true, up to a factor of two, when the integrals over $\triangle^n$ are replaced by integrals over a compact $K \subset \triangle^n$ for which $\pi^{-1}(K)$ is geodesically-convex. Here, $\pi : \mathbb{CP}^n \to \overline{\triangle^n}$ is the moment map. In the case where $n = 1$, the condition on K means that K is connected, contains one of the endpoints of the interval $\triangle^1$, and is contained in one of the halves of the interval $\triangle^1$.

Remark 4.7 Assumption (47) and even the more precise condition (48) seem a bit strict. We suspect that this is the fault of the hasty transition from (37) to (38) above. Perhaps a more subtle analysis, in the spirit of Barthe and Cordero-Erausquin [5], may transform the strict condition (47) into a parameter incorporated in the resulting Poincaré-type inequality.

Remark 4.8 Theorem 4.3 and its generalization Theorem 4.5 essentially follow by analyzing the Fubini-Study metric on $\mathbb{CP}^n$. It seems that there is a developed theory of "canonical" Kähler metrics on certain toric manifolds, and in many cases we even have an everywhere non-negative Ricci form. Our limited understanding of this theory has so far prevented us from extracting additional meaningful Poincaré-type inequalities.

5 From the Orthant to the Full Space

In this section we deduce Theorem 1.3 from Theorem 1.1 and from some essentially known facts. We say that an unconditional $\rho : \mathbb{R}^n \to \mathbb{R}$ is increasing when the restriction $\rho|_{\mathbb{R}^n_+}$ is

increasing. We say that it is decreasing when $x \mapsto -\rho(x)$ is increasing. The following lemma begins our analysis of the finite-dimensional space of functions on $\mathbb{R}^n$ that are constant on each orthant. Recall the definition (6) of the H^{-1} norm of a function.

Lemma 5.1 *Let $R > 0$, and let μ be the uniform probability measure on the interval $[-R, R]$. Suppose $f(x) = \mathrm{sgn}(x) = x/|x|$ for $x \neq 0$. Then,*

$$\|f\|_{H^{-1}(\mu)} \leq \frac{R}{\sqrt{3}} = \sqrt{\int_{\mathbb{R}} x^2 d\mu(x)}. \tag{50}$$

Proof: Integrating by parts, we see that for any smooth function g,

$$\begin{aligned}\frac{1}{2R}\int_{-R}^{R} fg &= \frac{1}{2R}\int_0^R [g(x) - g(-x)]\, dx = \frac{1}{2R}\int_0^R (R - x)\left(g'(x) + g'(-x)\right) dx \\ &\leq \frac{1}{2R}\sqrt{\int_0^R (R-x)^2 dx \int_0^R |g'(x) + g'(-x)|^2\, dx} \leq \frac{1}{2R}\sqrt{\frac{2R^3}{3}\int_{-R}^{R} |g'(x)|^2\, dx},\end{aligned}$$

where we used the Cauchy-Schwartz inequality. The bound (50) now follows from the definition (6) of the H^{-1}-norm. □

Suppose $\rho : \mathbb{R} \to \mathbb{R}$ is a probability density that is unconditional (i.e., even) and decreasing. It is elementary to verify that there exists a probability measure λ on $[0, \infty)$, such that

$$\rho(x) = \int_0^\infty \left(\frac{1_{[-R,R]}(x)}{2R}\right) d\lambda(R) \qquad \text{(for almost every } x \in \mathbb{R})$$

where $1_{[-R,R]}$ is the characteristic function of the interval $[-R, R]$. From Lemma 2.2 and Lemma 5.1 we conclude that for any probability measure μ on $\mathbb{R}$ with an unconditional, decreasing density,

$$\|\mathrm{sgn}(x)\|_{H^{-1}(\mu)} \leq \sqrt{\int_{\mathbb{R}} x^2 d\mu(x)}. \tag{51}$$

Note that when ρ is an unconditional, decreasing function on $\mathbb{R}^n$, the restriction of ρ to any line parallel to one of the axes, is a one-dimensional unconditional, decreasing function. From (51) and Lemma 2.2 we therefore obtain the following:

Corollary 5.2 *Suppose μ is a probability measure on $\mathbb{R}^n$ with an unconditional, decreasing density. Let $\ell = 1, \ldots, n$, and suppose that $f : \mathbb{R}^n \to \{-1, 1\}$ is a measurable function which does not depend on the ℓ^{th} coordinate. Set*

$$g(x) = f(x)\mathrm{sgn}(x_\ell) \qquad \textit{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then,

$$\|g\|_{H^{-1}(\mu)} \leq \sqrt{\int_{\mathbb{R}^n} x_\ell^2 d\mu(x)}.$$

Let $G = \{-1,1\}^n \cong (\mathbb{Z}/(2\mathbb{Z}))^n$, a commutative group with 2^n elements, where

$$xy = (x_1 y_1, \dots, x_n y_n) \qquad \text{for } x, y \in \{-1,1\}^n.$$

Denote by $\mathcal{H}$ the space of functions $f : G \to \mathbb{R}$ with $\sum_{x \in G} f(x) = 0$. For $x, y \in G$ and $f \in \mathcal{H}$ denote $T_x f(y) = f(xy)$. Suppose that we have two Hilbertian norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the space $\mathcal{H}$, with the property that

$$\|f\|_j = \|T_x f\|_j \tag{52}$$

for any $x \in G, f \in \mathcal{H}$ and $j = 1, 2$. From elementary representation theory, the supremum

$$\sup_{0 \neq f \in \mathcal{H}} \|f\|_1 / \|f\|_2$$

must be attained for a non-constant character $f : G \to \mathbb{R}$.

Lemma 5.3 *Suppose μ is a probability measure on $\mathbb{R}^n$ with an unconditional, decreasing density. Let $\mathcal{S} \subset L^2(\mu)$ be the finite-dimensional space spanned by functions f that are constant on orthants. That is, functions f such that*

$$f(x_1, \dots, x_n)$$

depends only on $sgn(x_1), \dots, sgn(x_n)$. Then, for any $f \in \mathcal{S}$ with $\int f^2 d\mu = 1$ and $\int f d\mu = 0$,

$$\|f\|^2_{H^{-1}(\mu)} \leq \max_{\ell = 1, \dots, n} \int_{\mathbb{R}^n} x_\ell^2 d\mu(x). \tag{53}$$

Proof: Denote by $\mathcal{H} \subset \mathcal{S}$ the subspace of all functions $f \in \mathcal{S}$ with $\int f d\mu = 0$, and consider the group $G = \{-1,1\}^n \cong (\mathbb{Z}/(2\mathbb{Z}))^n$. The linear space $\mathcal{H}$ is identified with the space of functions on G that sum to zero, since each of the 2^n orthants is identified with an element of G in an obvious manner. Furthermore, the $H^{-1}(\mu)$ norm and the $L^2(\mu)$ norm are both G-invariant Hilbertian norms on $\mathcal{H}$ in the sense of (52). It is therefore sufficient to verify (53) for non-constant characters, that is, for functions $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = \prod_{j=1}^n sgn(x_j)^{\delta_j} \qquad (x \in \mathbb{R}^n)$$

for some $0 \neq (\delta_1, \dots, \delta_n) \in \{0,1\}^n$. Note that all of these characters are of the form

$$f(x) = g(x) sgn(x_\ell)$$

for some $\ell = 1, \dots, n$ and for some measurable function $g : \mathbb{R}^n \to \{-1,1\}$ which does not depend on x_ℓ. Corollary 5.2 therefore applies, and implies (53). $\square$

Proof of Theorem 1.3: By applying a linear transformation of the form

$$\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto (\sqrt{V_1} x_1, \dots, \sqrt{V_n} x_n) \in \mathbb{R}^n$$

we reduce matters to the case $V_1 = \ldots = V_n = 1$. We will consider the norms corresponding to the expressions appearing on the right-hand side of (2) and of (3). That is, for a locally Lipschitz function $g \in L^2(\mu)$ set

$$\|g\|_{P^1(\mu)}^2 = \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{k^2}{k-1} x_i^2 \left|\partial^i g(x)\right|^2 d\mu(x),$$
$$\|g\|_{Q^1(\mu)}^2 = \int_{\mathbb{R}^n} \sum_{i=1}^n \left(\frac{k^2}{k-1} x_i^2 + 1\right) \left|\partial^i g(x)\right|^2 d\mu(x).$$

Then

$$\|g\|_{Q^1(\mu)}^2 = \|g\|_{P^1(\mu)}^2 + \|g\|_{H^1(\mu)}^2 \tag{54}$$

where $\|g\|_{H^1(\mu)}^2 = \int |\nabla g|^2 d\mu$. The dual norms are defined, for $f \in L^2(\mu)$, via

$$\|f\|_{P^{-1}(\mu)} = \sup_{\|g\|_{P^1(\mu)} \neq 0} \frac{\int f g d\mu}{\|g\|_{P^1(\mu)}}, \quad \|f\|_{Q^{-1}(\mu)} = \sup_{\|g\|_{Q^1(\mu)} \neq 0} \frac{\int f g d\mu}{\|g\|_{Q^1(\mu)}},$$

where the suprema run over all locally Lipschitz functions $g \in L^2(\mu)$. Using a standard duality argument we deduce from (54) that for any $f_1, f_2 \in L^2(\mu)$,

$$\|f_1 + f_2\|_{Q^{-1}(\mu)}^2 \leq \|f_1\|_{P^{-1}(\mu)}^2 + \|f_2\|_{H^{-1}(\mu)}^2 \tag{55}$$

whenever the right-hand side is finite. In order to prove (3), it suffices to show that for any $f \in L^2(\mu)$ with $\int f d\mu = 0$,

$$\|f\|_{Q^{-1}(\mu)} \leq \|f\|_{L^2(\mu)}. \tag{56}$$

(Strictly speaking, this will imply (3) only for a locally Lipschitz $f \in L^2(\mu)$, yet the generalization to a locally Lipschitz $f \in L^1(\mu)$ is simple, as is explained at the proof of Theorem 1.1 above). For $f : \mathbb{R}^n \to \mathbb{R}$ and $\delta \in \{-1, 1\}^n$ denote

$$f_\delta(x) = f(\delta_1 x_1, \ldots, \delta_n x_n) \qquad \text{for } x \in \mathbb{R}_+^n.$$

We write $\mathcal{G} \subseteq L^2(\mu)$ for the subspace of all $f \in L^2(\mu)$ which satisfy

$$\int_{\mathbb{R}_+^n} f_\delta d\mu = 0 \qquad \text{for all } \delta \in \{-1, 1\}^n.$$

Suppose that $g \in L^2(\mu)$ is a locally Lipschitz function with

$$\|g\|_{P^1(\mu)}^2 = \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{k^2}{k-1} x_i^2 \left|\partial^i g(x)\right|^2 d\mu(x) \leq 1. \tag{57}$$

For $\delta \in \{-1, 1\}^n$ let $E_\delta \in \mathbb{R}$ be such that $\int_{\mathbb{R}^n} (g_\delta - E_\delta) d\mu = 0$. According to (57) and to Theorem 1.1,

$$\sum_{\delta \in \{-1,1\}^n} \int_{\mathbb{R}_+^n} (g_\delta - E_\delta)^2 d\mu \leq 1.$$

Consequently, for any $f \in \mathcal{G}$,

$$\int_{\mathbb{R}^n} fg d\mu = \sum_{\delta \in \{-1,1\}^n} \int_{\mathbb{R}^n_+} f_\delta g_\delta d\mu = \sum_{\delta \in \{-1,1\}^n} \int_{\mathbb{R}^n_+} f_\delta (g_\delta - E_\delta) d\mu$$
$$\leq \sqrt{\sum_{\delta \in \{-1,1\}^n} \int_{\mathbb{R}^n_+} f_\delta^2 d\mu} \cdot \sqrt{\sum_{\delta \in \{-1,1\}^n} \int_{\mathbb{R}^n_+} (g_\delta - E_\delta)^2 d\mu} \leq \sqrt{\int_{\mathbb{R}^n} f^2 d\mu}.$$

We thus proved that

$$\|f\|_{P^{-1}(\mu)} \leq \|f\|_{L^2(\mu)} \qquad \text{for any } f \in \mathcal{G}. \tag{58}$$

Next, observe that $\mathcal{G}$ is the orthogonal complement to the subspace $\mathcal{S}$ from Lemma 5.3. Fix $f \in L^2(\mu)$ with $\int f d\mu = 0$. Then f may be represented as $f = g + s$, where $g \in \mathcal{G}, s \in \mathcal{S}$ and $\int s d\mu = 0$. From (55), (58) and Lemma 5.3,

$$\|f\|^2_{Q^{-1}(\mu)} \leq \|g\|^2_{P^{-1}(\mu)} + \|s\|^2_{H^{-1}(\mu)} \leq \|g\|^2_{L^2(\mu)} + \|s\|^2_{L^2(\mu)} = \|f\|^2_{L^2(\mu)},$$

and the desired (56) is proven. The "Furthermore" part of the theorem follows immediately from Theorem 1.1. $\square$

6 A direct approach for the orthant

In this section we provide another proof of Theorem 1.1, which does not involve spaces of twice the dimension. We prove the following slight generalization of Theorem 1.1, see also Remark 2.9.

Theorem 6.1 *Let $n \geq 1$. Let $k_1, \ldots, k_n > 1$ be real numbers, not necessarily integers. Suppose that μ is a Borel measure on $\mathbb{R}^n_+$ with density $\exp(-\varphi)$, where $\varphi : \mathbb{R}^n_+ \to \mathbb{R}$ is a smooth function such that*

$$\mathbb{R}^n_+ \ni (x_1, \ldots, x_n) \mapsto \varphi\left(x_1^{k_1}, \ldots, x_n^{k_n}\right)$$

is a convex function on $\mathbb{R}^n$. Assume that $f : \mathbb{R}^n_+ \to \mathbb{R}$ is a μ-integrable, locally Lipschitz function with $\int f d\mu = 0$. Then,

$$\int_{\mathbb{R}^n_+} f^2 d\mu \leq \int_{\mathbb{R}^n_+} \sum_{i=1}^n \frac{k_i^2}{k_i - 1} x_i^2 \left|\partial^i f(x)\right|^2 d\mu(x). \tag{59}$$

Proof: For $x \in \mathbb{R}^n_+$ we denote here

$$\pi(x) = (\pi_1(x), \ldots, \pi_n(x) = (x_1^{k_1}, \ldots, x_n^{k_n}).$$

Then $\varphi(\pi(x))$ is a convex function. Set

$$\psi(x) = \varphi(\pi(x)) - \sum_{i=1}^{n} (k_i - 1) \log x_i \qquad\qquad (x \in \mathbb{R}_+^n).$$

Since $\varphi(\pi(x))$ is convex, its Hessian is positive semi-definite. Therefore,

$$\left\langle \left(\nabla^2 \psi(x)\right)^{-1} U, U \right\rangle \leq \sum_{i=1}^{n} \frac{x_i^2}{k_i - 1} |U^i|^2$$

for any $x \in \mathbb{R}_+^n$ and $U = (U^1, \dots, U^n)$. From the Brascamp-Lieb inequality [7, Theorem 4.1], we conclude that for any locally Lipschitz function $f : \mathbb{R}_+^n \to \mathbb{R}$,

$$\text{(60)} \qquad \int_{\mathbb{R}_+^n} f e^{-\psi} = 0 \qquad \Rightarrow \qquad \int_{\mathbb{R}_+^n} f^2 e^{-\psi} \leq \int_{\mathbb{R}_+^n} \sum_{i=1}^{n} \frac{x_i^2}{k_i - 1} |\partial^i f(x)|^2 e^{-\psi(x)} dx.$$

Equivalently, for any locally Lipschitz function $f : \mathbb{R}_+^n \to \mathbb{R}$ with

$$\int_{\mathbb{R}_+^n} f(x) \left(\prod_{i=1}^{n} x_i^{k_i - 1} \right) e^{-\varphi(\pi(x))} dx = 0,$$

we have

$$\text{(61)} \quad \int_{\mathbb{R}_+^n} f^2 \left(\prod_{i=1}^{n} x_i^{k_i - 1} \right) e^{-\varphi(\pi(x))} dx \leq \int_{\mathbb{R}_+^n} \sum_{i=1}^{n} \frac{x_i^2}{k_i - 1} |\partial^i f|^2 \left(\prod_{i=1}^{n} x_i^{k_i - 1} \right) e^{-\varphi(\pi(x))} dx.$$

Observe that $\prod_{i=1}^{n} k_i x_i^{k_i - 1}$ is precisely the Jacobian determinant of π. Furthermore, if $f(x) = g(\pi(x))$, then

$$x_i \partial^i f(x) = k_i \pi_i(x) \partial^i g(\pi(x)).$$

From (61) we see that for any locally Lipschitz $f : \mathbb{R}_+^n \to \mathbb{R}$ with $\int f e^{-\varphi} = 0$,

$$\int_{\mathbb{R}_+^n} f^2 e^{-\varphi(x)} dx \leq \int_{\mathbb{R}_+^n} \sum_{i=1}^{n} \frac{k_i^2}{k_i - 1} x_i^2 |\partial^i f|^2 e^{-\varphi(x)} dx.$$

$\square$

Theorem 6.1 immediately implies the corresponding refinements of Corollary 1.2 and Theorem 1.3, as described in the Introduction.

Remark 6.2 We currently do not know of any direct approach for Theorem 1.4 or even for the Poincaré inequalities obtained for the simplex in Section 4. Still, we cannot escape the feeling that the symmetries we produce by adding extra dimensions are somewhat artificial. Perhaps we are overlooking a direct method, that could lead to simpler proofs and generalizations of the results in this manuscript.

References

[1] Abreu, M., *Kähler geometry of toric manifolds in symplectic coordinates.* Symplectic and contact topology: interactions and perspectives. Fields Inst. Commun., 35, Amer. Math. Soc., Providence, RI, (2003), 1-24.

[2] Abreu, M., *Kähler metrics on toric orbifolds.* J. Differential Geom., 58, no. 1, (2001), 151-187.

[3] Anttila, M., Ball, K., Perissinaki, I., *The central limit problem for convex bodies*. Trans. Amer. Math. Soc., 355, no. 12, (2003), 4723–4735.

[4] Bakry, D., Émery, M., *Diffusions hypercontractives* (French). Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., 1123, Springer, Berlin, (1985), 177-206.

[5] Barthe, F., Cordero-Erausquin, D., *Invariances in variance estimates*. Preprint. Available under `http://arxiv.org/abs/1106.5985`

[6] Bobkov, S. G., *On concentration of distributions of random weighted sums.* Ann. Prob., 31, no. 1, (2003), 195–215.

[7] Brascamp, H. J., Lieb, E. H., *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation.* J. Funct. Anal., 22, no. 4, (1976), 366-389.

[8] Brezis, H., *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[9] Cannas da Silva, A., *Lectures on Symplectic Geometry*. Lecture Notes in Math., 1764, Springer-Verlag, 2008.

[10] Chiang, Y.-J., *Harmonic Maps of V-Manifolds*. Ann. Global Anal. Geom., Vol. 8, No. 3 (1990), 315–344.

[11] Diaconis, P., Freedman, D., *Asymptotics of graphical projection pursuit.* Ann. Statist., 12, no. 3, (1984), 793–815.

[12] Eldan, R., Klartag, B., *Approximately gaussian marginals and the hyperplane conjecture*. Proc. of a workshop on "Concentration, Functional Inequalities and Isoperimetry", Contermporary Math., 545, Amer. Math. Soc., (2011), 55–68.

[13] Escobar, J., *Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate.* Comm. Pure Appl. Math., 43, no. 7, (1990), 857-883.

[14] Folland, G. B., *Introduction to Partial Differential Equations*. Mathematical Notes, Princeton University Press, Princeton, NJ, 1976.

[15] Gromov, M., *Convex sets and Kähler manifolds.* Advances in differential geometry and topology, World Sci. Publ., Teaneck, NJ, (1990), 1-38.

[16] Klartag, B., *A central limit theorem for convex sets.* Invent. Math. 168, no. 1, (2007), 91-131.

[17] Klartag, B., *A Berry-Esseen type inequality for convex bodies with an unconditional basis.* Probab. Theory Related Fields 145, no. 1-2, (2009), 1-33.

[18] Klartag, B., *High-dimensional distributions with convexity properties.* Proc. of the Fifth Euro. Congress of Math., Amsterdam, July 2008. Eur. Math. Soc. publishing house, (2010), 401–417.

[19] Matskewich, T., Sobolevskii, P., *The best possible constant in generalized Hardy's inequality for convex domain in* $\mathbb{R}^n$. Nonlinear Anal. 28, no. 9, (1997), 1601-1610.

[20] Müller, C., *Spherical harmonics.* Lecture Notes in Math., 17, Springer-Verlag, Berlin-New York, 1966.

[21] Petersen, P., *Riemannian geometry.* Second edition. Graduate Texts in Mathematics, 171. Springer, New York, 2006.

[22] Sudakov, V. N., *Typical distributions of linear functionals in finite-dimensional spaces of high-dimension.* (Russian) Dokl. Akad. Nauk. SSSR, 243, no. 6, (1978), 1402–1405. English translation in Soviet Math. Dokl., 19, (1978), 1578–1582.

[23] Tian, G., *Canonical metrics in Kähler geometry.* Notes taken by Meike Akveld. Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag, Basel, 2000.

[24] Villani, C., *Topics in optimal transportation.* Graduate Studies in Mathematics, 58. American Mathematical Society, Providence, RI, 2003.

[25] von Weizsäcker, H., *Sudakov's typical marginals, random linear functionals and a conditional central limit theorem*. Probab. Theory and Related Fields, 107, no. 3, (1997), 313–324.